Optimal cap on pension contributions

by

András Simonovits

September 18, 2012

Institute of Economics, CERS, Hungarian Academy of Sciences
also Mathematical Institute of Budapest University of Technology,
and Department of Economics of CEU
Budapest, Budaörsi út 45, Hungary
e-mail: simonovits.andras@krtk.mta.hu, telephone: 36-1-325-5582, fax: 36-1-319-3631

Abstract*

In our model, the government operates a mandatory proportional (contributive) pension system to substitute for the low life-cycle savings of the low-paid myopes. The socially optimal contribution rate is high (equalizing young- and old-age consumption for them), while an appropriate cap on pension contributions makes room for the saving of high-paid far-sighted workers. In our parameterization (with a discrete as well as a Pareto earning distribution), the optimal cap can be determined but its aggregate impact is negligible.

Keywords: pensions, contribution rate, contribution cap, maximum for taxable earnings

JEL Numbers: H53, H24

* I express my gratitude to N. Barr, Zs. Cseres-Gergely, H. Fehr, J. Köllö, G. Körösi and Th. Matheson (my discussant at the IIPF Congress in Dresden held in 2012) for friendly comments. This research has received generous financial support from OTKA K 67853.
1. Introduction

Since the publication of World Bank (1994), the debate on privatization and prefunding of the unfunded public pension systems has focused attention on the socially optimal choice of the contribution rates to the arising two pillars. In terms of gross wage, the public contribution rate varies across time and space dramatically: while the US social security contribution rate is as low as 12.4 percent, the Hungarian rate is as high as 34 percent. These differences are due to differences in the breakdown to the employer’s and the employee’s rates, the dependency rates (the ratios of pensioners to the workers), the replacement rates (the ratios of average benefits to averages wages), the degree of redistribution and the share of the private pillar in the total pension system.

Much less attention has been paid to the socially optimal choice of the earnings cap, officially called the maximum for taxable earnings (for a summary statistics, see Table 1 in Valdés-Prieto and Schwarzhaupt 2011). Such a cap (or ceiling) implies an upper limit on the mandatory pension contributions as well as on the future benefits. While the contribution rate affects every worker, a well-designed cap only influences the best-paid; nevertheless, the cap also deserves attention. We give only two examples to illustrate poor design of the cap. (i) In the 1950s, in Great Britain the cap was fixed at the minimum wage (probably about half the average wage), transforming the usually earning-related contributions into flat ones. It was only realized much later that such a solution reduces excessively the flat benefit and then was replaced by a much higher cap, making the rise of the benefit and the redistribution possible within the public system. (ii) In Hungary, the ratio of the cap to the average gross wage sank from 3.3 to 1.6 between 1992 and 1996 just to grow from 1.6 to 3.1 between 1997 and 2005. (Note also that the cap on the contributions to the tax-favored voluntary pension system is much lower and it interacts with the mandatory one; Simonovits, 2011.)

Most experts see the role of the cap as a hidden personal income tax. For example, in some countries, the cap only applies to the employee’s contribution and the benefit, therefore from an economic point of view the employer’s contribution above the cap is a pure personal income tax. (In Hungary, the former contribution rate is 10 percent of the gross wages, while the latter is 24 percent. The uncapped contributions alone provide 6 percent of the total pension contributions and would provide 10 percent of the de facto personal income tax.) Others justify the existence of the cap as follows: the government has no mandate to force high old-age consumption on high-earners and removing the cap would further increase the perverse redistribution from the poor to the rich caused by the strong correlation between lifetime earning and life expectancy.

To present a third role of the cap in the mandatory public system we apply two widely accepted idealizations: (1) the higher the wage, the higher is the discount factor and (2) any dollar saved privately rather than in a public pension system raises the old-age benefit more. Then the third role can be formulated as follows: the capped pension contribution ensures sufficient mandatory savings for the low-earning short-sighted but leave sufficient room for the more efficient voluntary savings for the high-earning far-sighted. We are aware that our idealizations are only approximately valid. Ad 1. There are low-paid who are far-sighted and there are high-paid who are short-sighted. Nevertheless, the correlation between discount factors and wages appears to be strongly positive, therefore the atypical combinations can be neglected. Ad 2) In fact, the voluntary (private) saving may also be less efficient than the mandatory (public)
Correspondingly we consider a very simple model, where workers only differ by their discount factors and wages but have the same age when they start working, retire or die. Using a social welfare function, where individual discounting is eliminated, a socially optimal pension system successfully combines these two tasks. (Note also that in an imaginary world, where the discount factor is a decreasing rather than an increasing function of the wage, the cap should be replaced by a floor.)

For the sake of simplicity, apart from the Appendix, we confine our analysis to proportional pension systems (and write contribution rather than pay-roll tax), where the benefit is proportional to the net covered wage. To avoid absurd results (suggesting the elimination of any mandatory pension system), we assume realistically that the frequency of lower-paid workers is so high, and their wages and their discount factors are so low that the socially optimal contribution rate is maximal, equalizing young and old-age consumption for the myopes. The cap reduces the effective contribution rate (i.e. the ratio of contribution to earning) of workers earning above the cap, making room for private savings. In addition to the contribution rate, the value of the optimal cap can also be determined, but in the most realistic version of the model, its aggregate impact is hardly noticeable. In other words: in a wide interval, when the cap is slightly raised, the marginal gains of the lower-paid and short-sighted workers are canceled by the marginal losses of the higher-paid and far-sighted workers.

Formulating our model, we have adopted Feldstein (1987)'s framework: evaluating the mandatory pension systems, he emphasized the role of different discount factors in the accumulation of voluntary life-cycle savings and used a paternalistic social welfare function for correcting individual myopia. As a short-cut, he neglected wage differences, consequently he confined his attention to flat or means-tested benefits. He found that the optimal means-tested system is typically welfare superior to the flat one, because it limits the benefit provision to the myopes and leaves room for more efficient private savings.

The present paper modifies Feldstein (1987) in several dimensions: (i) as is already mentioned, different discounting types have also different wages, namely higher earners discount the utility of old-age consumption less than lower earners do, (ii) the pension benefit is proportional rather than flat, (iii) contributions and benefits are capped. We have also made three technical modifications: (a) the length of the time spent in retirement is shorter than instead of being equal to that in work, therefore consumption is understood as intensity (consumption per unit period); (b) the lifetime utility function is of a discounted Leontief rather than usual Cobb–Douglas-type (but see Appendix); (c) the wage–discount factor distribution can be not only discrete but also continuous.

Issue (b) deserves some explanation. As is known, the additive discounted utility function introduced by Samuelson (1937) contains the discounted Cobb–Douglas and the undiscounted Leontief utility functions as special cases. But following the spirit of Shane–Loewenstein–O’Donoghue (2002), we give up the additivity assumption, and introduce a discounted Leontief utility function. In our specification, the two-period utility function is the minimum of the discounted young-age consumption and of the undiscounted old-age consumption. It is easy to see that the lower the discount factor,
the lower is the optimal old-age consumption. By the way, this specification eliminates
the need for paternalistic social welfare function.

Recently, Valdés-Prieto and Schwarzhaupt (2011) created a model to analyze the
issue of the optimal coercion including the choice of cap. They considered a larger set
of errors of judgments on remaining life span and future needs, furthermore, modeled
various pension systems. They put the value of the optimal cap near the 80th percent
of the earning distribution.

To allude to other approaches, we only mention just one paper (Casamatta, Cremer
and Pestieau, 2000) which allowed for mixed (partly flat, partly proportional) systems
and replaced social welfare maximization with the median voter’s self-optimization of
the contribution rate.

Note that models of this type neglect real-life complications like growth, inflation
and population aging. Therefore, in our static model, we cannot consider the dynamic
problem of carving out a private pillar from a public one (Diamond and Orszag, 2004
vs. Feldstein, 2005). Neither can we evaluate proposals like Diamond and Orszag
(2004) who would phase-in a 3 percent tax on incomes above the cap to reduce the
long-term imbalance of the US Social Security. The problem of time-inconsistency in
private savings (e.g. Laibson, 1997) is also out of scope. (We do not follow Docquier
(2002), who replaced the representative generation’s social welfare function by a proper
one, comprising the life-time utilities of all present and future generations.)

To help the reader and also demonstrate the robustness of our results, we study four
versions of the model in a row (in increasing level of sophistication) analytically and
illustrate some of them numerically. In the single-type version of the model, there is
no role for the cap and the socially optimal contribution rate is either zero (for a suffi-
ciently far-sighted population) or maximal (for an insufficiently far-sighted population):
equalizing the young- and old-age consumptions. In the multi-type versions, widespread
and strong myopia calls for a high contribution rate.

In the two-type version of the model, the role of the cap is to reduce the effective
pension contribution rate for the higher-paid far-sighted type, putting the cap at the
lower wage. (The end result resembles the British flat contribution–flat benefit sys-
tem in the 1950s mentioned above.) In the three-type version of the model, the first
(lowest-earning) type is strongly myope, the second (medium-earning) type is moder-
ately and the third (highest-earning) type is hardly myope. Their frequencies are steeply
diminishing. The inevitably high contribution rate seriously undermines even the sav-
ing incentives of the medium type. Therefore the socially optimal cap must also fully
cover this type. But to ensure some room for saving by the high-earners, the govern-
ment excludes any further increase in the cap. The socially optimal cap is equal to the
medium wage, which may be greater than the average wage. (Note that our medium
wage generally differs from the median wage, which is typically lower than the average
wage.)

Introducing a continuous-type model, the choice of the socially optimal cap becomes
a mathematically intriguing problem. We prove a quite complex optimality condition:
marginally raising the cap above the optimum, the marginal gain of subcritical earners
(with zero savings) is equal to the marginal loss of supercritical earners (with positive
savings). We have a simple corollary: the optimal cap is a decreasing function of the
efficiency of the private savings. For the Pareto-2 distribution (where the minimum wage
is just half the average wage and the wage variance is infinite)—the socially optimal contribution rate remains the maximal one—and the socially optimal cap is between 3–4 times the average wage, covering 97–98 percent of the workers. Note, however, that the social welfare is practically insensitive to the cap.

We are uncertain in what degree is this result only due to the omissions of the model or reflects deeper economic relations. For example, in the two cases relegated to an Appendix, our results change dramatically. a) Allowing for redistribution without efficiency losses, the proportional pension system might be replaced by a flat one, excluding any cap. b) Replacing our discounted Leontef-type utility function by a discounted Cobb-Douglas-one, a strongly progressive social welfare function or very low discount factors are needed to reestablish a sensible pension system with a sensible cap.

Further research is needed to generalize the results for wider setting. We have already alluded to the role of our specific utility function, the lack of redistribution or the progressivity of the social welfare function. Deterministic and stochastic changes in the relative earnings position during one’s lifetime may also be important. The dependence of type-specific life expectancy and private saving’s efficiency on earning is also important in practice, and may call for progressive benefits.

The structure of the remainder is as follows. Section 2 presents the model, Section 3 displays numerical illustrations. Finally, Section 4 concludes. Appendix contains the modifications involving linear benefit functions and Cobb-Douglas-utility functions.

2. Model

First we shall outline a framework, and then we shall specify versions of the model with various number of types.

Framework

We consider a very simple pension model, where workers only differ in wages and discount factors but wages do not vary with age. A type can be described by his total wage $w$ and his discount factor $\delta$, the joint distribution function of $(w, \delta)$ by $F$ and the corresponding expectations by $E$. For example, the average wage is normalized as $Ew = 1$. Workers pay contributions $\tau w$ up to $\tau\bar{w}$, where $0 < \tau < 1$ is the contribution rate to the mandatory pension system and $\bar{w}$ is the wage ceiling or cap. Hence his covered total wage and net wage are respectively

$$\hat{w} = \min(w, \bar{w}) \quad \text{and} \quad v = w - \tau\hat{w}.$$ 

Calculating pension benefits, a related variable is used, the net covered earning:

$$\hat{v} = (1 - \tau)\hat{w}.$$ 

Note that for wages lower than or equal to the cap, its existence is indifferent; for wages higher than the cap: $w > \bar{w}$, the cap ensures that the effective contribution rate is $\hat{\tau} = \tau\bar{w}/w < \tau$, reducing the original contribution rate for workers earning above the cap. The function of the cap is as follows: lower-paid myopic workers can be locked into
a pension system with a high contribution rate, but higher-paid far-sighted workers pay a lower effective rate.

By assumption, every pension benefit is proportional (to the net covered wages): 
\[ b = \beta \hat{v}. \] Everybody works for a unit period, and everybody spends the a shorter period in retirement with a common length \( \mu, 0 < \mu \leq 1. \) Hence the pension balance is simply 
\[ \mu \beta E \hat{v} = \tau E \hat{w}, \] i.e. \( \mu \beta = \tau (1 - \tau). \) Since the two periods’ lengths are different, we use intensities, i.e. quantities per a unit time period even if it is not always mentioned.

In addition to paying pension contributions, workers can also privately save for old-age: \( s \geq 0. \) Denoting the compound interest factor by \( \rho \geq 1, \) the intensity of the decumulated saving is \( \mu \frac{1}{\rho} s. \)

We can now describe the young and old-age consumption (intensities):
\[ c = v - s \quad \text{and} \quad d = b + \mu^{-1} \rho s. \]

To determine the individually optimal savings, we introduce a discounted non-additive lifetime Leontief utility function:
\[ U(w, \delta, c, d) = \min(\delta c, d), \]
where \( \delta \) is the discount factor, \( 0 \leq \delta \leq 1. \) At first sight it may be strange that the young- rather than the old-age consumption is discounted by \( \delta \) but note that at any reasonable optimum with positive saving, \( \delta c^* = d^*, \) hence the lower the \( \delta, \) the lower is \( d^* \) and the higher is \( c^*. \) (With zero saving, \( \delta c > d, \) thus the discount factor does not play any role.)

We must separate two cases: either 1) nonnegative saving intention or 2) negative saving intention. At the optimum of type 1, \( \delta c = d, \) hence equation
\[ \delta v - \delta s = b + \mu^{-1} \rho s \]
yields the optimal saving intention:
\[ s^i = \frac{\delta v - b}{\delta + \mu^{-1} \rho} \]
which materializes if it is positive or zero: \( \delta v \geq \mu^{-1} \tau \hat{w}. \)

Hence the optimal worker’s consumption is
\[ c^*_1 = \frac{\rho v + \mu b}{\mu \delta + \rho}. \]

On the other hand, if \( s^i < 0, \) then \( s^*_2 = 0, \) hence \( c^*_2 = v \) and \( d^*_2 = b. \)

Analyzing the indirect utility function, we shall drop the notational distinction between the two regimes:
\[ u(w, \delta) = \min[\delta c^*, d^*]. \]

By choosing the contribution rate \( \tau \) and the wage cap \( \hat{w}, \) the government maximizes the expected value of the paternalistic, undiscounted indirect utility functions, i.e.
\[ V(\tau, \hat{w}) = E u^*(w, 1) = E \min[c^*, d^*]. \]
We shall see that this distinction is indifferent, because the discounted as well as the undiscounted utility function is equal to \( d^* \).

Since the utility is linear in consumption, our social welfare function has a direct economic meaning: if a system produces 1% higher social welfare than the other, then the original wages need to be scaled up by 1% in the second system to compensate for suboptimality.

Next we study the single-, the two- and the three-type models and then we turn to the most demanding continuous-type model.

### A single-type model

We start the analysis with a single-type model, where the typical individual has a wage \( w > 0 \) and a discount factor \( \delta, 0 \leq \delta \leq 1 \). Here the cap has no function but the model is a useful ground for preparation.

Using the general formulas above, we can deduce the saving function

\[
s^* = \frac{\delta(1 - \tau)w - \beta(1 - \tau)w}{\delta + \mu^{-1} \rho} \geq 0 \quad \text{if} \quad 0 < \tau \leq \tau_\delta,
\]

where \( \tau_\delta \) is the maximal contribution rate which allows for nonnegative saving intention.

An optimal contribution rate obviously cannot be higher than the absolute maximum:

\[
\bar{\tau} = \frac{1}{1 + \mu^{-1}} = \tau_1,
\]

because such a rate would suboptimally raise the old-age consumption above the young-age consumption: \( d > c \). Indeed, \( (1 - \tau)w \leq \mu^{-1} \tau w \) implies \( \tau \leq \bar{\tau} \)—to be assumed from now on.

To determine the socially optimal \( \tau \), we need to separate the two cases and to do so, introduce the critical discount factor

\[
\delta^* = \frac{1}{1 + \mu(1 - \rho^{-1})}.
\]

Note that if the private saving is efficient, i.e. \( \rho > 1 \), then \( 0 < \delta^* < 1 \), which is always assumed from now on.

We shall prove the following theorem.

**Theorem 1.** In a single-type model with efficient private savings, no cap is needed and the socially optimal contribution rate is maximal for subcritical discount factors and zero for supercritical discount factors and both maximal and zero for the critical discount factor:

\[
\tau^o = \begin{cases} 
\bar{\tau} & \text{if } 0 \leq \delta < \delta^*; \\
0, \bar{\tau} & \text{if } \delta = \delta^*; \\
0 & \text{if } \delta^* < \delta \leq 1.
\end{cases}
\]

**Remark.** In the limit, when the private saving is as efficient as the mandatory pension system, i.e. \( \rho = 1 \), then \( \delta^* = 1 \) and the indeterminacy becomes total, extending to the entire interval \([0, \bar{\tau}]\): a perfect substitution arises between mandatory contributions and voluntary savings.
Proof. To find the optimal $\tau$, we must separate the two cases of nonnegative and negative saving intentions.

ad 1) Substituting $v = (1 - \tau)w$ and $b = \mu^{-1}\tau w$ into the formula $c_1 = d_1/\delta$, we obtain

$$d_1(\tau) = \frac{\delta [\rho - (\rho - 1)\tau] w}{\mu \delta + \rho}.$$ 

It is easy to see that $d_1(\tau)$ is a decreasing function in the interval $[0, \bar{\tau}]$, thus its maximum is reached at $\tau = 0$, where

$$d_1(0) = \frac{\delta \rho w}{\mu \delta + \rho}.$$ 

ad 2) Using the benefit formula $d_2(\tau) = \mu^{-1}\tau w$, $d_2(\tau)$ is obviously an increasing function in the interval $[0, \bar{\tau}]$, and its maximum is reached at $\bar{\tau}$:

$$d_2(\bar{\tau}) = \frac{w}{1 + \mu} = c_2(\bar{\tau}).$$

To decide which maximum is higher, one must compare $d_1(0)$ and $d_2(\bar{\tau})$. A simple calculation yields the value of the critical discount factor $\delta^*$ which equates the two quantities:

$$\frac{\delta^* \rho}{\mu \delta^* + \rho} = \frac{1}{1 + \mu}.$$ 

For $\delta < \delta^*$, $d_1(0) < d_2(\bar{\tau})$ and $\tau^* = \bar{\tau}$ hold. For $\delta > \delta^*$, $d_1(0) > d_2(\bar{\tau})$ and $\tau^* = 0$ hold. For $\delta = \delta^*$, $d_1(0) = d_2(\bar{\tau})$ holds and $\tau^*$ is indeterminate: either 0 or $\bar{\tau}$.

We can now formulate a trivial

**Corollary.** For multi-type models, a) if all discount factors are lower than the critical one, then the socially optimal contribution rate and the corresponding cap are equal to the maxima; b) if all discount factors are higher than the critical one, then the socially optimal contribution rate is zero, while the corresponding cap is indifferent.

Note that both cases cover quasi-single type populations.

A two-type model

Our single (or quasi-single)-type model is not only unrealistic but by construction excludes the study of optimal cap. Therefore we shall continue with a two-type model: low and high types (denoted by subscript L and H) with wages $0 < w_L < w_H$ and discount factors $0 \leq \delta_L < \delta_H \leq 1$, respectively. To avoid trivialities covered in the Corollary above, we assume that the critical discount factor separates the two discount factors:

$$0 \leq \delta_L < \delta^* < \delta_H \leq 1,$$

suggesting a maximal contribution rate for type L and a minimal (in fact, zero) contribution rate for type H. For practical reasons, such a solution is politically impossible to legislate, therefore as a second-best solution, the government might introduce a cap on the contribution, just at the low wage. To aggregate the type-effects, we define a social welfare function. First we introduce the frequencies of both types, to be denoted by $f_L > 0$ and $f_H > 0$ with $f_L + f_H = 1$. We assume a utilitarian social welfare function,

$$V(\tau, \bar{w}) = f_L d_L + f_H d_H.$$

First we formulate the optimality condition of that arrangement in a very simple setup.
Theorem 2. Suppose two types with $0 < w_L < w_H$ and $\delta_L = 0$ and $\delta_H = 1$. If the parameters satisfy condition

$$\frac{f_L}{f_H} > \frac{\mu(\rho - 1)}{\mu + \rho},$$

then the socially optimal contribution rate is maximal: $\tau^o = \bar{\tau}$ and the corresponding cap is equal to the lower wage: $\bar{w}^o = w_L$.

Proof. In this extreme setup, $d_L(\tau)$ is increasing and $d_H(\tau)$ is decreasing in the entire interval $[0, \bar{\tau}]$. Assuming that $\bar{w} \leq w_H$, $d_L(\tau) = \mu - (\rho - 1)\bar{w}\tau$ holds and

$$d_H(\tau) = \frac{\rho - (\rho - 1)\bar{w}\tau}{\mu + \rho},$$

where $\bar{\tau} = \frac{\bar{\tau}}{w_H}$. The social welfare function is

$$V(\tau, \bar{w}) = f_L w_L\tau + f_H \rho w_H - (\rho - 1)\bar{w}\tau.$$ 

Evidently the lower the cap, the higher is the social welfare, therefore $\bar{w}^o = w_L$. Then the coefficient of $w_L\tau$ in $V(\tau, \bar{w})$ is equal to

$$\frac{f_L}{\mu} - \frac{f_H(\rho - 1)}{\mu + \rho}$$

and this should be positive to have an increasing function of the tax rate $\tau$.

Using extreme assumptions on discounting (à la Feldstein), the displayed condition basically presumes that the efficiency of the private saving cannot be too high or the share of low earners too low. Specifically, if $f_L > f_H$ holds, then the condition is true for any $\rho$. It is time to relax the extreme conditions of full myopia and full far-sightedness. To sharpen Theorem 2, we introduce another critical discount factor, also depending on the ratio of the low to the high wage. Notation:

$$\delta^*_{LH} = \frac{1}{1 + \mu(1 - \rho^{-1})(1 - w_L/w_H)}.$$ 

It is easy to see that $\delta^* < \delta^*_{LH} < 1$ and in the limit case of $w_L = 0$, $\delta^*_{LH} = \delta^*$.

To avoid absurd results, we assume that the frequency of lower-paid is so high, their wage and discount factor are so low that the socially optimal contribution rate is maximal: $\tau = \bar{\tau}$, yielding $c_L = d_L = w_L/(1 + \mu)$.

We present the generalization of Theorem 2.

Theorem 3. In a two-type model, we assume a subcritical low discount factor: $0 < \delta_L < \delta^*_{LH}$. Then the socially optimal cap is maximal for subcritical high discount factors and minimal for supercritical high discount factors:

$$\bar{w}^o = \begin{cases} w_H & \text{if } \delta_H < \delta^*_{LH}; \\ w_L & \text{if } \delta_H > \delta^*_{LH}. \end{cases}$$

Remark. For $\delta_H = \delta^*_{LH}$, there is indeterminacy: any $\bar{w} \in [w_L, w_H]$ is optimal.
Proof. Type H’s paternalistic utility function needs some discussion. The cap is only useful if the effective contribution rate \( \tau \) yields higher old-age consumption \( d_H \) than does \( \bar{\tau} \). This is only possible if \( s_H^0 > 0 \). Looking at the formulas:

\[
d_{1H}(\bar{\tau}) = \frac{\delta_H[\rho w_H - (\rho - 1)\bar{\tau}w_L]}{\mu\delta_H + \rho} > \frac{w_H}{1 + \mu} = d_{2H}(\bar{\tau}).
\]

Rearranging the inequality,

\[
(1 + \mu)\rho w_H \delta_H - \mu(\rho - 1)w_L \delta_H - \mu w_H \delta_H > \rho w_H.
\]

Using the definition of \( \delta_{LH}^* \), this last inequality is equivalent to the second branch of the optimality condition. In the other case, the opposite holds.

The two-type model is quite illuminating but it gives extreme values; either a too low cap: \( w_L \) or a too high cap: \( w_H \). We need to introduce a third, medium type, to obtain a nontrivial socially optimal cap.

A three-type model

We shall work with the following three-type model: low, medium and high types with wages \( w_L < w_E < w_H \) and discount factors \( 0 \approx \delta_L < \delta_E < \delta_H \approx 1 \). Now

\[
V(\tau, \bar{w}) = f_Ld_L + f_Ed_E + f_Hd_H.
\]

The optimal cap does not rise above the medium wage \( w_E \), because then its only function would be punish type H’s saving. On the other hand, we know that the optimal cap does not stay below the minimal wage \( w_L \), because it would not relieve any type. Hence the only relevant interval for the optimal cap is \( w_L < \bar{w} \leq w_E \).

Now we have the following ‘scheme’: \( v_E = w_E - \tau\bar{w} \) and \( \hat{v}_E = (1 - \tau)\bar{w} \), while \( v_H = w_H - \tau\bar{w} \) and \( \hat{v}_H = (1 - \tau)\bar{w} \). We must ensure that in the relevant interval, \( s_E^0 < 0 \).

At this stage we have only a conjecture rather than a theorem.

Conjecture. In our three-type model, we have wages \( 0 < w_L < w_E < w_H \) and discount factors \( 0 \approx \delta_L < \delta_E < \delta_H \approx 1 \). For certain moderate assumptions (most individuals are quite myopic and low-paid), the socially optimal contribution rate is again maximal: \( \tau^o = \bar{\tau} \) and the related cap is equal to the medium wage: \( \bar{w}^o = w_E \).

Continuous distributions

Finally we turn to the more realistic continuous wage and discount factor distributions. Since typically the higher the earning, the higher is the discount factor, thus we may assume that the joint distribution of \( (\delta, w) \) is one-dimensional. Let the discount factor \( \delta(w) \) be a continuous and monotone increasing function of the wage in the interval \( [w_m, w_M] \) with \( 0 < w_m < w_M \leq \infty \), which in turn has a positive density function \( f \) and a corresponding distribution function

\[
F(w) = \int_{w_m}^{w} f(\omega) d\omega
\]
with \( F(w_m) = 0 \) and \( F(w_M) = 1 \).

Furthermore, \( \delta_m = \delta(w_m) \) and \( \delta_M = \delta(w_M) \), \( 0 \leq \delta_m < \delta_M \leq 1 \). To avoid trivialities, we make the following two assumptions.

A1. For the lowest wage \( w_m \), the corresponding discount factor \( \delta_m \) is so low and the value of the density function \( f(w_m) \) is so high that the lowest-paid needs to pay full contribution: \( \tau^o = \bar{\tau} \).

A2. For the highest wage \( w_M \), the highest discount factor \( \delta_M \) is so high and the wage ratio \( w_m/w_M \) is so low that \( \delta_M > \delta^*_mM \) (Theorem 3), i.e. even for the maximal contribution rate, there are some room for private saving: \( \bar{\tau}^o < w_M \).

We are looking for conditions which guarantee the existence of at least one cap \( \bar{\tau} \) such that leaves room for private savings. First we define the concept of the critical wage \( w^* \) for any cap \( \bar{\tau} \in (w_m, \bar{w}M) \) and it is an increasing function of the cap.

**Lemma 1.** Under A1–A2, i.e. if the maximal discount factor \( \delta_M \) is sufficiently high:

\[
\delta_M(w_M - \bar{\tau}w_m) > \mu^{-1} \bar{\tau}w_m,
\]

then there exists a maximal cap \( \bar{\tau}_M \) such that there exists a critical wage \( w^* \) for any cap \( \bar{\tau} \in (w_m, \bar{\tau}_M) \) and it is an increasing function of the cap.

**Remarks.** 1. Recall that \( \bar{\tau} = 1/(1+\mu^{-1}) \). If \( \delta_M = 1 \), then our condition is simply \( w_M > w_m \). Otherwise, \( w_M/w_m \) and \( \delta_M \) together should be large enough to satisfy the condition.

2. For the limiting case of constant discount factor \( \delta \), the critical wage is a simple multiple of the cap:

\[
w^* = (1 + \mu^{-1} \delta^{-1}) \bar{\tau} \bar{w} \geq (1 + \mu^{-1}) \bar{\tau} \bar{w} = \bar{w}.
\]

**Proof.** Denote the left hand side of the critical wage equation by \( g(\cdot) \) which is clearly increasing. Furthermore, \( g(\bar{w}) = \delta(\bar{w})(1-\bar{\tau})\bar{w} - \mu^{-1} \bar{\tau} \bar{w} < 0 \) and by our condition, \( g(w_M) > 0 \). By Bolzano’s theorem, there exists a critical \( w^* \).

Replacing the one-variable function \( g(w^*) \) by a two-variable function \( G(\bar{w}, w^*) \), the implicit function theorem yields

\[
w^*(\bar{w}) = \frac{G_2}{G_1} = \frac{[\delta(w^*) + \mu^{-1}] \bar{\tau}}{\delta'(w^*)(w^* - \bar{\tau} \bar{w}) + \delta(w^*)} > 0,
\]

where \( G_1 \) and \( G_2 \) are the respective partial derivatives.

The main problem of the optimal choice of the cap is that for the maximal contribution rate, raising the cap in the interval \( (w_m, \bar{w}_M) \), lower-paid’s old-age consumption increases, but higher-paid’s old-age consumption decreases. What is the socially optimal balance? To answer this question we need the following notation (to avoid confusion
with the differential operator \(d\), we replace here the lower-case letter by its upper-case counterpart:

\[
D(w, \bar{w}) = \max[d_1(w, \bar{w}), d_2(w, \bar{w})],
\]

where (copying from the single-type model)

\[
d_1(w, \bar{w}) = \frac{\delta(w)[\rho w - (\rho - 1)\bar{\tau}\bar{w}]}{\mu\delta(w) + \rho} \quad \text{and} \quad d_2(w, \bar{w}) = \frac{\bar{\tau}\bar{w}}{\mu}.
\]

The social welfare function is defined as

\[
V(\bar{w}) = \int_{w_m}^{w_M} D(w, \bar{w}) f(w) \, dw.
\]

We shall prove

**Theorem 4.** Under the assumptions of Lemma, in a continuous-type model with the maximal contribution rate \(\bar{\tau}\), the locally optimal cap \(\bar{w}^o\) satisfies the following condition:

\[
F(w^*) - F(\bar{w}^o) = (\rho - 1) \int_{w^o}^{w_M} \frac{\mu\delta(w)}{\mu\delta(w) + \rho} f(w) \, dw,
\]

where

\[
\delta(w^*) = \mu^{-1}\bar{\tau}\bar{w}.
\]

**Remark.** Intuitively, our optimality condition states that at a marginal rise of the optimal cap, the marginal gain of subcritical earners \((\bar{w}^o < w < w^*)\) is equal to the marginal loss of supercritical earners \((w^* < w < w_M)\), those earning below the cap are not affected. It is another question whether there exists such a cap and if there exists, then it gives a maximum or not.

**Proof.** We shall cut the interval \([w_m, w_M]\) into three, therefore

\[
V(\bar{w}) = \int_{w_m}^{\bar{w}} d_2(w, \bar{w}) f(w) \, dw + \int_{\bar{w}}^{w^*} d_2(w, \bar{w}) f(w) \, dw + \int_{w^*}^{w_M} d_1(w, \bar{w}) f(w) \, dw.
\]

If the social welfare function has a local maximum, then its derivative is equal to zero. Inserting the formulas for \(d_1\) and \(d_2\) into \(V\), omitting the unessential interval \([w_m, \bar{w}]\) and canceling the positive derivative of the upper limit of the first parametric integrals and the negative derivative of the lower limit of the second, provides the necessary condition:

\[
V'(\bar{w}) = \int_{\bar{w}}^{w^*} \frac{\bar{\tau} f(w)}{\mu} \, dw - \int_{w^*}^{w_M} \frac{\delta(w)(\rho - 1)\bar{\tau}}{\mu\delta(w) + \rho} f(w) \, dw = 0.
\]

A simple transformation yields the optimality condition.

**A3.** The share \(F(w^*(\bar{w})) - F(\bar{w})\) of subcritical earners (who earn above the cap but do not save) decreases with the cap. (We shall see that this assumption holds for any constant discount factor and for our parametric distribution named after Wilfredo Pareto.)
**Corollary.** Under the assumptions of Theorem 4 and A3, the optimal cap is a decreasing function of the private saving efficiency.

**Proof.** Surprisingly, the critical earning is independent of the efficiency. Denote the right hand side of the condition of Theorem 4 as

\[ R(\rho, \bar{w}) = \int_{w^*}^{w_m} \frac{(\rho - 1)\mu\delta(w)}{\mu\delta(w) + \rho} f(w) \, dw. \]

For a fixed cap, raising \( \rho \), increases the integrand, hence increases \( R \). By A3, the left hand side, \( F(w^*(\bar{w})) - F(\bar{w}) \) can only increase if the cap \( \bar{w} \) decreases. \( \blacksquare \)

**Pareto-distribution**

For analytical reasons, we use the *Pareto-distribution* (cf. Diamond and Saez, 2011) with density function

\[ f(w) = \sigma w^\sigma w^{1-\sigma} \quad \text{for} \quad w > w_m, \]

where \( \sigma > 1 \) is the exponent of the distribution and \( w_m \) is the minimum wage. It is easy to give an explicit formula for the distribution function:

\[ F(w) = \int_{w_m}^{w} f(\omega) \, d\omega = 1 - w_m^\sigma w^{-\sigma} \quad \text{for} \quad w \geq w_m. \]

Hence \( F(w_m) = 0 \) and \( F(\infty) = 1 \), and its expectation can explicitly be calculated:

\[ Ew = \int_{w_m}^{\infty} w f(w) \, dw = \frac{\sigma w_m}{\sigma - 1}. \]

Since we normalized the expectation as unity, the minimum wage is given as

\[ w_m = \frac{\sigma - 1}{\sigma}. \]

In practice, \( \sigma \approx 2 \), then \( w_m \approx 1/2 \). We also display the variance of the Pareto-distribution:

\[ Ew^2 = \frac{\sigma w_m^2}{\sigma - 2} = \frac{(\sigma - 1)^2}{\sigma(\sigma - 2)} \quad \text{for} \quad \sigma > 2. \]

As a detour, we mention that for our distribution function and with a fixed discount factor \( \delta \), A3 used in Corollary holds:

\[ F(w^*(\bar{w})) - F(\bar{w}) = \{1 - [(1 + \mu^{-1}\delta^{-1})\bar{\tau}]\}\bar{w}^{-\sigma} \]

is a decreasing function of cap \( \bar{w} \).

For our unlimited distribution, let \( w_N \) be the maximal value at which the wage distribution is cut and we represent the wages above \( w_N \) by a cleverly chosen \( w_K \). Since the censored expected wage is

\[ E\min(w, w_N) = 1 - \frac{w_m^\sigma w_N^{-\sigma+1}}{\sigma - 1}, \]

12
therefore
\[ 1 = \int_{w_m}^{w_N} wf(w) \, dw + [1 - F(w_N)]w_K. \]

Hence
\[ w_K = \frac{\sigma}{\sigma - 1} w_N. \]

For example, for \( \sigma = 2 \), the representative highest wage is double of the “maximum”: \( w_K = 2w_N \).

### 3. Numerical illustrations

Since our problem is quite involved, therefore we turn to numerical illustrations. When we apply ‘very’ discrete rather than continuous distributions, then our calculations have only limited numerical use.

Before proceeding, let us recall that unlike others (Feldstein, 1987) we have distinguished the lengths of the working and of the retirement periods. Denoting their ratio by \( \mu \) rather than 1, we receive more realistic numbers. For example, our socially maximal contribution rate \( \bar{\tau} = 1/(1+\mu^{-1}) \) drops from 1/2 to 1/3 as we replace \( \mu = 1 \) by 0.5. On the other hand, if we took into account that the socially optimal discount factor is less than one, then we could further reduce the contribution rate further, even to 1/4.

We shall choose an annual efficiency of saving \( \rho_1 = 1.0233 \) for private savings yielding a compound 30-year efficiency factor \( \rho = 2 \), and search for the socially optimal contribution rate and the cap.

#### Discrete distributions

We start with discrete distributions. Note that for the single-type distribution, for \( \rho = 2 \), the critical discount factor \( \delta^* = 0.8 \) (annually \( \delta^*_1 = 0.9926 \)). In the two-type case, the lower limit on \( f_H/(1 - f_H) \) of Theorem 2 is quite mild: 0.2, i.e. \( f_H > 0.16 \).

For an illustration, we shall work with three types, with the following frequencies: \( f_L = 0.7 \), \( f_E = 0.25 \) and \( f_H = 0.05 \), wage rates \( w_L = 0.7 \), \( w_E = 1.3 \) and \( w_H = 1.7 \), when \( \delta^*_{EH} = 0.944 \). The following 30-year compound discount factors were used: \( \delta_L = 0.215 \), \( \delta_E = 0.545 \), and \( \delta_H = 0.97 \), reducing to annual discount factors \( \delta_{1L} = 0.977 \), \( \delta_{1E} = 0.988 \), and \( \delta_{1H} = 0.999 \). Having found the approximate socially optimal values \( \tau^o = 0.333 \) (rather than the exact 1/3) and \( \bar{w}^o = w_E \), Table 1 displays the total and net wages, the covered net wages, savings, and the two consumption pairs.
Table 1. The optimal outcome for 3 types

<table>
<thead>
<tr>
<th>Type</th>
<th>Total $w$</th>
<th>Net wage $v$</th>
<th>Net covered $\tilde{v}$</th>
<th>Saving $s^o$</th>
<th>Worker’s consumption $c^o$</th>
<th>Pensioner’s consumption $d^o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>0.7</td>
<td>0.467</td>
<td>0.467</td>
<td>0.000</td>
<td>0.467</td>
<td>0.466</td>
</tr>
<tr>
<td>E</td>
<td>1.3</td>
<td>0.867</td>
<td>0.867</td>
<td>0.000</td>
<td>0.867</td>
<td>0.866</td>
</tr>
<tr>
<td>H</td>
<td>1.7</td>
<td>1.267</td>
<td>0.867</td>
<td>0.073</td>
<td>1.194</td>
<td>1.159</td>
</tr>
</tbody>
</table>

Remark. $\tau^o = 0.333$, $\bar{w}^o = 1.3$.

Table 2 confirms our Conjecture: only type H saves, and apart from numerical rounding-off errors, the low and the medium pairs of intensities are equal: $d_i \approx c_i$, $i = L, E$, while $d_H = \delta_H c_H$ would be assured.

Table 2. The impact of cap on old-age consumption

<table>
<thead>
<tr>
<th>Cap $\bar{w}$</th>
<th>Medium $d^o_E$</th>
<th>High $d^o_H$</th>
<th>Social welfare $V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.666</td>
<td>1.198</td>
<td>0.626</td>
</tr>
<tr>
<td>1.1</td>
<td>0.733</td>
<td>1.185</td>
<td>0.641</td>
</tr>
<tr>
<td>1.2</td>
<td>0.799</td>
<td>1.172</td>
<td>0.655</td>
</tr>
<tr>
<td>1.3</td>
<td>0.866</td>
<td>1.159</td>
<td>0.670</td>
</tr>
<tr>
<td>1.4</td>
<td>0.866</td>
<td>1.146</td>
<td>0.668</td>
</tr>
<tr>
<td>1.5</td>
<td>0.866</td>
<td>1.133</td>
<td>0.666</td>
</tr>
<tr>
<td>1.6</td>
<td>0.866</td>
<td>1.120</td>
<td>0.664</td>
</tr>
<tr>
<td>1.7</td>
<td>0.866</td>
<td>1.132</td>
<td>0.666</td>
</tr>
</tbody>
</table>

Remark. See Table 1, $d^o_L = 0.466$.

Note that if $\delta_{1H}$ were diminished by 0.003 to 0.996, then the optimal cap would jump from $w_E = 1.3$ to $w_H = 1.7$ and $c_H = 1.214$ and $d_H = 1.077$ would yield a lower utility than the previous optimum.

Pareto distribution

To give a flavor of the behavior of the Pareto distribution, we display the values of the distribution function and the covered expected earnings for selected values of caps, appearing below. Note how fast the probability converges to 1 as the relative value of the cap goes to 4, and how slowly the share of the covered earnings does so. For example, 1.6% of the highest earners still have 12.5% of the total earnings.
Table 3. Pareto-probabilities and covered earnings for varying caps

| Earning cap $\bar{w}$ | Probability $F(\bar{w})$ | Share of covered earnings $P(w < \bar{w})E(w|w < \bar{w})$ |
|------------------------|--------------------------|--------------------------------------------------|
| 0.707                  | 0.500                    | 0.250                                             |
| 1.0                    | 0.750                    | 0.500                                             |
| 1.5                    | 0.889                    | 0.667                                             |
| 2.0                    | 0.938                    | 0.750                                             |
| 2.5                    | 0.960                    | 0.800                                             |
| 3.0                    | 0.972                    | 0.833                                             |
| 4.0                    | 0.984                    | 0.875                                             |

**Remark.** $\sigma = 2.$

Recall that in our model, the discount factor is an increasing function of the wage: $\delta = \delta(w)$. To map an infinite interval into a finite one, we assume

$$\delta(w) = (\delta_m - \delta_M)e^{\xi(w_m-w)} + \delta_M,$$

where $\xi > 0$ stands for the strength of the dependence. Note that for finite $w_N$, $\delta(w_N) < \delta_M$, but for high $w_N/w_m$, the error is small. We shall work with $\delta_m = 0.25$, $\delta_M = 1$ and $\xi = 0.5$.

We shall divide the interval $[w_m, w_M]$ into $n = 100$ subintervals such a way that the division points $w_i$ form a geometrical sequence: and at integration, the representative points are the geometrical means of the subsequent points: $w_{i+1} = qw_i$ and $z_i = \sqrt{w_iw_{i+1}}$. The mass of the remaining infinite part is $1 - F(w_M) = 0.0001$ (with $w_M = 50$) and the earning $w_K = 100$ represents the average highest wage.

As before, the optimal contribution rate is again maximal: $\tau^o = \bar{\tau}$. According to Table 4, the social welfare function is hardly sensitive to the ratio of cap to average wage in the interval $[2.5, 4]$. Even after multiplying the social welfare function by 10, the variation only appears in the last digit. On the other hand, for a contribution rate slightly below the maximum: $\tau = 0.3$, the dependence is stronger and $\bar{w}^o = 2.5$. With such a flat social welfare function, there is not much practical use in Theorem 4!
Table 4. The welfare impact of the contribution rate and the cap

<table>
<thead>
<tr>
<th>Contribution rate $\tau$</th>
<th>Cap $\bar{w}$</th>
<th>$10\times$ Social welfare $10U$</th>
<th>Expected saving $E_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.300</td>
<td>2.5</td>
<td>6.157</td>
<td>0.019</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>6.154</td>
<td>0.016</td>
</tr>
<tr>
<td></td>
<td>3.5</td>
<td>6.146</td>
<td>0.014</td>
</tr>
<tr>
<td></td>
<td>4.0</td>
<td>6.137</td>
<td>0.013</td>
</tr>
<tr>
<td></td>
<td>4.5</td>
<td>6.129</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>6.122</td>
<td>0.011</td>
</tr>
<tr>
<td>0.333</td>
<td>2.5</td>
<td>6.690</td>
<td>0.017</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>6.704</td>
<td>0.015</td>
</tr>
<tr>
<td></td>
<td>3.5</td>
<td>6.709</td>
<td>0.013</td>
</tr>
<tr>
<td></td>
<td>4.0</td>
<td>6.709</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td>4.5</td>
<td>6.708</td>
<td>0.010</td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>6.705</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Remark. $\rho = 2$.

Changing the exponent of the wage distribution $\sigma$ does not change the results qualitatively, only the cap rises. The influence of the efficiency $\rho$ on the welfare is hardly visible, but its rise diminishes the optimal value of the cap (Corollary to Theorem 4). The rise of annual private efficiency from 1.02 to 1.04 and then to 1.06 diminishes the *approximate* socially optimal relative value of cap from 3.5 to 3 to 2.5 but the welfare impact is very weak (Table 5). Using the data from Table 3, our optimal cap covers a much higher share of the workers than that of Valdés-Prieto and Schwarzhaupt: 97–98 vs. 80 percent. What is surprising is that at the same time, the expected value of saving diminishes from 0.014 to 0.01 to 0.007. The income effect dominates the substitution effect.

Table 5. The impact of efficiency on the optimal cap

<table>
<thead>
<tr>
<th>Annual efficiency $\rho_1$</th>
<th>Cap $\bar{w}$</th>
<th>$10\times$ Social welfare $10U$</th>
<th>Expected saving $E_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.02</td>
<td>3.5</td>
<td>6.700</td>
<td>0.014</td>
</tr>
<tr>
<td></td>
<td>4.0</td>
<td>6.701</td>
<td>0.013</td>
</tr>
<tr>
<td>1.04</td>
<td>3.0</td>
<td>6.748</td>
<td>0.010</td>
</tr>
<tr>
<td></td>
<td>3.5</td>
<td>6.746</td>
<td>0.009</td>
</tr>
<tr>
<td>1.06</td>
<td>2.5</td>
<td>6.782</td>
<td>0.007</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>6.782</td>
<td>0.006</td>
</tr>
</tbody>
</table>

Remark. $\tau^o = 0.333$. 

16
Finally, Table 6 explores the impact of the minimal discount factor on the optimal value of the cap. As is to be expected, the optimal cap is a diminishing function of the minimal discount factor: raising $\delta_m$ from 0 to 0.5, the optimal cap decreases from 4.5 to 3, with slightly lifting the social welfare and the saving.

<table>
<thead>
<tr>
<th>Minimal discount factor $\delta_m$</th>
<th>Relative cap $\bar{w}$</th>
<th>$10\times$ social welfare $10U$</th>
<th>Expected saving $E_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>4.5</td>
<td>6.703</td>
<td>0.010</td>
</tr>
<tr>
<td>0.25</td>
<td>3.5</td>
<td>6.709</td>
<td>0.013</td>
</tr>
<tr>
<td>0.50</td>
<td>3.0</td>
<td>6.721</td>
<td>0.015</td>
</tr>
</tbody>
</table>

Remark. $\rho = 2$

In the Appendix, we shall shortly discuss two other important modifications: the impact of introducing a redistributive pension system and replacing the discounted Leontief utility function with a Cobb–Douglas utility function and a paternalistic social welfare function.

4. Conclusions

We have constructed a very simple model of the proportional pension system to analyze the socially optimal contribution rate and especially the contribution cap. We have concentrated on the contradiction between the needs of low-earning myopic and high-earning far-sighted types: the former need a high contribution rate to make up for their low savings; the latter need a low contribution rate to make room for their high and efficient savings. A politically convenient compromise is the introduction of an appropriate cap on the contribution: the well-chosen cap does not diminish the contribution as well as the welfare of the myopes but relieves the far-sighted from a part of the contribution burden.

This model is just the beginning. It neglects very important issues: the heterogeneity of the life spans and of the efficiency of private savings. This may suggest the introduction of progressive pension systems, for example, the proportional part is complemented by a uniform basic benefit. Then the analysis of the progressive personal income tax also comes to the fore. The flexibility of the labor supply and the underreporting of the true labor income are other important issues, have been studied in other papers.
Appendix

Redistributive benefits

In this Subsection, we introduce redistributive linear benefit function: the proportional part is supplemented by a flat part: \( b = \alpha + \beta \hat{v} \). Hence the pension balance is simply \( \mu (\alpha + \beta \mathbf{E} \hat{v}) = \tau \mathbf{E} \hat{w} \). We skip the theoretical discussion and only present a generalization of the calculations of Table 4 from proportional to linear benefits. Since the model neglects the negative impact of redistribution on the labor supply, small wonder that the socially optimal system is characterized by maximal contribution rate \( \tau = 1/3 \), flat benefit: \( \beta = 0 \) and lack of cap: \( \bar{w}^o = \infty \). The realistic interaction of redistribution and efficiency needs further investigations. Here we only display the dependence of the flat benefit and the resulting social welfare on the cap varying between 3 and 5.

<table>
<thead>
<tr>
<th>Cap ( \bar{w} )</th>
<th>Flat benefit ( \beta )</th>
<th>10x Social welfare ( 10U )</th>
<th>Expected saving ( \mathbf{E}s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.593</td>
<td>7.036</td>
<td>0.028</td>
</tr>
<tr>
<td>4</td>
<td>0.612</td>
<td>7.152</td>
<td>0.026</td>
</tr>
<tr>
<td>5</td>
<td>0.623</td>
<td>7.226</td>
<td>0.025</td>
</tr>
</tbody>
</table>

Remark. \( \rho = 2 \) and \( \beta = 0 \).

Cobb–Douglas-utility functions

In this Subsection, we replace Leontief with the usual discounted Cobb–Douglas utility function:

\[
U(c, d) = \log c + \mu \delta \log d.
\]

Repeating the previous calculations, the optimal saving intention is now to be determined from

\[
\frac{1}{v - s} = \frac{\delta \rho}{b + \mu^{-1} \rho s}.
\]

With simple algebra,

\[
s^i = \frac{\delta \rho v - b}{(\delta + \mu^{-1}) \rho}
\]

e tc.

Substituting the previous parameter values to the new formulas, we find that under the paternalistic social welfare function, the socially optimal pension system is no pension system at all: \( \tau^o = 0! \) However, if we diminish the minimal discount factor from 0.25 to 0, then we receive a socially optimal contribution rate what is maximal and the corresponding cap is slightly below 1: Table 4b.
Table 4.b. The welfare impact of the contribution rate and the cap for CD u.f.

<table>
<thead>
<tr>
<th>Cap $\bar{w}$</th>
<th>10× Social welfare $10U$</th>
<th>Expected saving $E_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>-8.914</td>
<td>0.084</td>
</tr>
<tr>
<td>0.6</td>
<td>-8.812</td>
<td>0.079</td>
</tr>
<tr>
<td>0.7</td>
<td>-8.776</td>
<td>0.075</td>
</tr>
<tr>
<td>0.8</td>
<td>-8.768</td>
<td>0.071</td>
</tr>
<tr>
<td>0.9</td>
<td>-8.772</td>
<td>0.068</td>
</tr>
</tbody>
</table>

Remark. $\rho = 2$, $\alpha = 0$ and $\delta_m = 0$, $\tau = 1/3$.

References


