An Application of Big Boss Games: 
The Reinsurance Market

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Abstract

Insurance games, and the so-called Reinsurance Games have already been considered in the literature (see Fragnelli and Marina (2003), Suijs et al (1997), Lemaire (2004) and Suijs et al (1999)). In this paper we introduce Reinsurance Games as Big Boss games (Muto et al, 1988). The policyholders are signing a contract with one insurer only, who is in charge of organising the proper insurance for the client. She is the boss in this game, therefore it is out of the question that she would be left out from the coalition. It is her choice whom she wants to include as reinsurer, therefore the role of the reinsurance companies is secondary.

In this paper we present a general algorithm for creating Reinsurance Games, and proving that an appropriate Big Boss Game can be found for each and every reinsurance situation. Moreover, we will show that for each Big Boss Game exists one (or more) Reinsurance Game, which can be associated with the original game.

Keywords: Insurance Games, Reinsurance Games, Big Boss Games

JEL Classification: C71, G22

1 Introducing Reinsurance Games

Why does a reinsurance company decide to share its risk, whereas she would be capable of insuring a given policy by herself? Is it really worth it to share both the risk and the profit?

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Consider a policy which requires a huge amount of money from the insurer in case of a given event. The probability of the accident occurring is very low, but in case it happens the insurer suffers a great loss. Therefore it seems a good idea to divide both the risk and the premium with other insurance companies.

What happens though if a company decides to participate in more than one insurance policies which are divided among other insurance companies? Isn’t it the same situation as keeping only one policy for his own? Although at first one can not tell the difference between these two situations, but the following example illustrates why sharing the risks should be a better solution.

1.1 An example

In the following pages we are going to introduce an example which illustrates why insurance companies should rely on each other. However, it should be noted, that the example simplifies reality, and some of the actuarial requirements for insurance will be ignored.

Let’s consider a market which consists of 4 insurance companies. One of them (Insurer no. 4) is thinking about accepting a contract which has a very low risk, but in case of an accident the payment is very high. From now on we will refer to this contract as the ‘big contract’. There are only two states of the world: one is when the insurer does not have to pay anything, and the other is where he is required to pay 1,200 units. The probability of the latter one is 1%.

In this example the premium of the big contract is given by its policyholder, which will be 270 units (from now on it will be denoted as $d$). The insurance companies have a price taker behaviour, therefore they are not trying to compete by offering lower premia. Their goal is to maximize their profit by lowering their costs, which makes them to follow both a competitive and a cooperative behaviour.

Insurer no. 4 has multiple choices: one is to reject the contract, one is to keep it to herself, or she also has the possibility of forming coalitions with the other insurers. She can either form a coalition with Insurer no. 1, or Insurer no. 2 or both, or she could ask Insurer no. 3 to join as well. However, the contract can not be accepted without Insurer no. 4, therefore all the coalitions, which do not contain her, are useless.

The three insurance companies are not identical. Insurer no. 1 possess 200 contracts, which each have a 10% chance of suffering a loss of 20 units, whereas Insurance no. 2 possess 400 of these contracts, Insurance no. 3 has 600 and Insurer no. 4 has 1600. Therefore these four companies have
different amounts as reserves. In this example we would like the companies to be solvent with a 99% probability.

Now consider the following: Insurer no. 4 decides not to accept the big contract, and she wants to be solvent with a 99% probability. Obviously, if the insurer wants to be absolutely safe, and be a 100% sure that she is capable of paying, then she should assume the worst: all the insurance events occurring. In reality this basically never happens, therefore the insurance company can keep lower amounts as reserves. How many losses will the company experience? No one can tell. But estimations can be made for the probability of the number of losses. This provides a tool for the insurer to calculate that with a 99% probability how much money she should pay.

Assuming that each ‘ordinary’ contract is independent from the others, we can assign a $\xi$ random variable with a categorical distribution to each of them. The distribution is the following for each $i=1,2,...,1600$:

$$P(\xi = 0) = 0.9$$
$$P(\xi = 20) = 0.1$$

Therefore, when we are seeking the lowest sufficient reserve, we are looking for the lowest $x$ to solve the following formula:

$$P\left(\sum_{i=1}^{x} \xi_i < x\right) = 0.99$$

The sum of $x_i$ follows a binomial distribution (as the sum of categorical distributions is always binomial), therefore the upper mentioned formula can also be considered as the inverse cumulative distribution function of a binomial distribution with the following parameters: $n = 1600$ and $p = 0.1$. It should also be remarked, that for continuous distributions we can find the exact amount where the cumulative distribution function equals to 0.99, therefore in these cases we are seeking $x$, which equals to the inverse cumulative distribution function at 0.99.

Solving the problem gives us $x = 3,740$, which means that if Insurer no. 4 keeps 3,740 units as reserve, then he has a 99% probability of being capable of paying for the policyholders’ losses.

If Insurer no. 4 decides to include the big contract in his portfolio as well, then it is obvious that he should keep a higher amount of reserve than before. From the big contract’s perspective there are only two states of the world: one is where the insurance event occurs, and one is where it doesn’t. Assuming that the big contract’s occurrence is independent from the ordinary
contracts another random variable $\eta$ with a categorical distribution can be introduced for it. The distribution is the following:

\[ P(\eta = 0) = 0.99 \]
\[ P(\eta = 1, 200) = 0.01 \]

Therefore, if Insurer no. 4 wants to include this contract in his portfolio he has to solve the following problem to calculate the new lowest sufficient reserve:

\[ P\left(\sum_{i=1}^{x} x_i + \eta < x\right) = 0.99 \]

In this case $x = 4,000$, which means that Insurer no. 4 has to increase his reserve by 260 units if he wants to include the big contract in his portfolio.

Now let’s examine Insurer no. 1, 2 and 3. In their cases the sufficient amount of reserve can be calculated the same way, but they obviously have different results. Table 1 illustrates how much each company should reserve only to insure their given contracts (from now on these will be denoted as $R^S_1$, where S denotes the coalitions), and the big risk as well (denoted by $R^S_2$). The difference between the two situations ($R^S_2 - R^S_1$) is also visible in the last column.

<table>
<thead>
<tr>
<th>Insurer</th>
<th>Reserve without big contract</th>
<th>Reserve with big contract</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>580</td>
<td>1,180</td>
<td>600</td>
</tr>
<tr>
<td>2</td>
<td>1,080</td>
<td>1,500</td>
<td>420</td>
</tr>
<tr>
<td>3</td>
<td>1,540</td>
<td>1,900</td>
<td>360</td>
</tr>
<tr>
<td>4</td>
<td>3,740</td>
<td>4,000</td>
<td>260</td>
</tr>
</tbody>
</table>

Table 1: Lowest sufficient reserves for singleton coalitions

The difference between the two situations can be considered as the cost of including the big contract into one’s portfolio. In the game theoretical context these differences can be assigned as the singleton coalitions’ costs.

What happens if the insurers decide to pool their resources? This is quite common behaviour when it comes to a contract which has low risk, but very high amount that should be paid in case of an insurance event occurs. In this example insurers will decide to pool all their ordinary contracts, and start to act as one big insurer. Although, complete share of all contracts is not a typical behaviour for insurers, but in this case we can calculate how much they could gain at most from the cooperation. If Insurer no. 1 and
no. 2 decide to form a coalition they will end up with $200 + 400 = 600$ ordinary contracts, therefore they will be as powerful as Insurer no. 3, who single-handedly possess 600 ordinary contracts. The possible coalitions, their lowest sufficient reserves in both cases and the difference among them can be seen in Table 2, where the calculations were based on the same methodology as before.

<table>
<thead>
<tr>
<th>Insurer</th>
<th>Reserve without big contract</th>
<th>Reserve with big contract</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>1,540</td>
<td>1,900</td>
<td>360</td>
</tr>
<tr>
<td>13</td>
<td>1,980</td>
<td>2,320</td>
<td>340</td>
</tr>
<tr>
<td>14</td>
<td>4,180</td>
<td>4,420</td>
<td>240</td>
</tr>
<tr>
<td>23</td>
<td>2,440</td>
<td>2,740</td>
<td>300</td>
</tr>
<tr>
<td>24</td>
<td>4,620</td>
<td>4,840</td>
<td>220</td>
</tr>
<tr>
<td>34</td>
<td>5,040</td>
<td>5,260</td>
<td>220</td>
</tr>
<tr>
<td>123</td>
<td>2,880</td>
<td>3,160</td>
<td>280</td>
</tr>
<tr>
<td>124</td>
<td>5,040</td>
<td>5,260</td>
<td>220</td>
</tr>
<tr>
<td>134</td>
<td>5,480</td>
<td>5,680</td>
<td>200</td>
</tr>
<tr>
<td>234</td>
<td>5,900</td>
<td>6,100</td>
<td>200</td>
</tr>
<tr>
<td>1234</td>
<td>6,340</td>
<td>6,520</td>
<td>180</td>
</tr>
</tbody>
</table>

Table 2: Lowest sufficient reserves for multiton coalitions

From this point it is easy to construct a game, where the 'differences' can be considered as the cost of each coalition. However, should be noted, that it was Insurer no. 4 who brought the big contract, therefore she can not be left out of any coalitions. Therefore for those coalitions that do not contain Insurer no. 4 the value of the coalition should be equal to 0 by definition.

Let $v(S)$ denote the value of each coalition, and $i^*$ the so-called Big Boss, who is represented by Insurer no. 4 in this example. The values of each coalition can be defined as the following:

$$v(S) = \begin{cases} 
\max \{ d - (R^S_2 - R^S_1), 0 \} & \text{if } i^* \in S \\
0 & \text{if } i^* \notin S
\end{cases}$$

If this formula is applied, then the costs of coalitions will be as in Table 3.

It has already been noted that in this example Insurer no. 1 and Insurer no. 2 together are exactly as powerful as Insurer no. 3 alone. Therefore it seems unnecessary for Insurer no. 4 to form a coalition with all of them, however he might also be indifferent regarding this question. Either the first two, or the third company could be excluded from this reinsurance business,
Table 3: The coalitions’ values

<table>
<thead>
<tr>
<th>S</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>123</th>
<th>124</th>
<th>134</th>
<th>234</th>
<th>1234</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(S)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4: Marginal contributions to the grand coalition

<table>
<thead>
<tr>
<th>Insurer</th>
<th>Marginal contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insurer no. 1</td>
<td>$v(1234) - v(234) = 20$</td>
</tr>
<tr>
<td>Insurer no. 2</td>
<td>$v(1234) - v(134) = 20$</td>
</tr>
<tr>
<td>Insurer no. 3</td>
<td>$v(1234) - v(124) = 40$</td>
</tr>
</tbody>
</table>

Table 5: Marginal contributions to the grand coalition

<table>
<thead>
<tr>
<th>Insurer</th>
<th>Marginal contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insurer no. 1</td>
<td>$v(124) - v(24) = 0$</td>
</tr>
<tr>
<td>Insurer no. 2</td>
<td>$v(124) - v(14) = 20$</td>
</tr>
</tbody>
</table>

and it will still be as profitable, or even better for Insurer no. 4 as it was when all the insurance companies were included. In this example the marginal contribution to the grand coalition’s value for each coalition can be seen in Table 4.

This means that Insurer no. 1 will demand at least 20 units from the profit, so does Insurer no. 2 and Insurer no. 3 would like to have 40 units at least, otherwise leaving the grand coalition.

Now let’s consider that Insurer no. 4 decides to include Insurer no. 1 and 2 only. In this case the marginal contributions to the grand coalition are as in Table 5.

In this scenario Insurer no. 1 seems to have no contribution to the grand coalition. Insurer no. 2 may demand 20, as she contributes to the grand coalition by that much.

Nevertheless, if Insurer no. 4 had decided to exclude both Insurer no. 1 and 2, and form a coalition with Insurer no. 3 only, then it would seem that the marginal contribution of Insurer no. 3 to the grand coalition equals 50-10=40. It is easy to see that combining the portfolios of Insurer no. 1 and no. 2 will have the same outcome as Insurer no. 3 by itself. However the marginal contributions of the first situation and the second are not equal. If
we assume that Insurers will later demand at least their contribution to the grand coalition, hence blackmailing the Big Boss, then the given game can not be stable. Therefore at least one Insurer must be eliminated from the game. The following formula makes sure that the above mentioned criteria holds:

\[ v(N) - v(S) \geq \sum_{i \in N \setminus S} v(N) - v(N \setminus \{i\}) \text{ if } i^* \in S \]

In this case the criteria is not satisfied. Table 6 represent the left and the right hand side of the inequality in the given case.

<table>
<thead>
<tr>
<th>Insurer</th>
<th>Left hand side</th>
<th>Right hand side</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>80</td>
<td>80</td>
<td>correct</td>
</tr>
<tr>
<td>14</td>
<td>60</td>
<td>60</td>
<td>correct</td>
</tr>
<tr>
<td>24</td>
<td>40</td>
<td>60</td>
<td>not correct</td>
</tr>
<tr>
<td>34</td>
<td>40</td>
<td>40</td>
<td>correct</td>
</tr>
<tr>
<td>124</td>
<td>40</td>
<td>40</td>
<td>correct</td>
</tr>
<tr>
<td>134</td>
<td>20</td>
<td>20</td>
<td>correct</td>
</tr>
<tr>
<td>234</td>
<td>20</td>
<td>20</td>
<td>correct</td>
</tr>
<tr>
<td>1234</td>
<td>0</td>
<td>0</td>
<td>correct</td>
</tr>
</tbody>
</table>

Table 6: Stability criteria

As it has been previously mentioned, in this situation Insurer no. 1 and Insurer no. 2 together contribute to the grand coalition more than the sum of their individual contributions it seems desirable to keep them in the game instead of Insurer no. 3. In this case the stability criteria holds (see in Table 7).

<table>
<thead>
<tr>
<th>Insurer</th>
<th>Left hand side</th>
<th>Right hand side</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>90</td>
<td>80</td>
<td>correct</td>
</tr>
<tr>
<td>14</td>
<td>60</td>
<td>60</td>
<td>correct</td>
</tr>
<tr>
<td>24</td>
<td>40</td>
<td>40</td>
<td>correct</td>
</tr>
<tr>
<td>124</td>
<td>0</td>
<td>0</td>
<td>correct</td>
</tr>
</tbody>
</table>

Table 7: Stability criteria without Insurer no. 3

If Insurer no. 3 gets excluded, then the Reinsurance Game will look like as Table 8 illustrates.
1.2 Preliminaries

In this paper we will show that reinsurance situations can be translated into the language of game theory. The so-called Reinsurance Game will be introduced, and a general construction methodology will also be provided. Our goal is to emphasize the connection between the Reinsurance Game and Big Boss Games. As the reinsurance situations require a distinguished player, called the Big Boss, it seems like all the Reinsurance Games are Big Boss Games as well, furthermore, all Big Boss Games can be associated with a Reinsurance Game. Further we will examine this.

The next section introduces some game theoretical definitions which are all applicable for the reinsurance situation. Then we will show the parallelism between the Reinsurance Games and Big Boss Games, starting with introducing the reinsurance model in general, followed by the construction of the Reinsurance Game, and then proving that this construction can be translated into a Big Boss Game. Conclusion follows.

2 Game Theoretical Background

In this section I would like to briefly review the most important definitions which of cooperative game theory, which are also applicable for Reinsurance Games.

**Definition 1** (Cooperative TU-Games).
A cooperative TU-game consists of a
(1) a finite set, \( N \), of \( n \) players,
(2) a value function \( v: 2^N \to \mathbb{R} \), i.e. for all \( S \subseteq N \), \( v(S) \) is a number that assigns the total utility of \( S \) if it forms a coalition and breaks off, and \( v(\emptyset) = 0 \).

As the example already showed, our goal is to find a cooperative TU-game for each reinsurance situation. This will later be described in general.

**Definition 2** (Monotone Games).
A \( v \) game is monotone if \( S \subseteq T \Rightarrow v(S) \leq v(T) \) for any \( S, T \subseteq N \).

Speaking about insurers this characteristic seems quite obvious. It simply means that by including one more reinsurer in the business will not decrease the sum of profit. Therefore the Reinsurance Game will always be monotone.
Definition 3 (Superadditive Games).
A game is superadditive if \( v(S) + v(T) \leq v(S \cup T) \) holds for any \( S, T \subseteq N \) such that \( S \cap T = \emptyset \).

For two disjoint groups of insurers it is always true that by pooling their portfolios they can get a profit which is greater or equal to the sum of the profit, that they could get separately. In this case, when only one of the above mentioned groups can contain the cedant, it is obvious that one of the groups will have a value of zero. Hence this definition for the given situation will be exactly as the previously described monotone game definition, which is why Reinsurance Games are always superadditive games.

Definition 4 (Big Boss Games).
Let \( v \) be a Monotone Game. \( v \) is called a Big Boss Game if there is one player, denoted by \( i^* \), satisfying the following two conditions:

- \( v(S) = 0 \) if \( i^* \notin S \),
- \( v(N) - v(S) \geq \sum_{i \in N \setminus S} v(N) - v(N \setminus \{i\}) \) if \( i^* \in S \).

Big Boss Games can also be associated with Reinsurance Games, however this requires presenting the exact definition of Reinsurance Games first. Therefore this statement will be discussed later.

3 Reinsurance Games and Big Boss Games
In the following pages we are going to present a reinsurance model, which is more general than the example, and which can be applied to various reinsurance situations.

3.1 The reinsurance model in general
Consider \( \{X_i\}_{i \in I} \) the set of ordinary contract types. In the previously described example there was only one kind of ordinary contract, which had a categorical distribution with a random variable of \( \xi \). Obviously each insurer might have different types of contracts, or let’s call them a group of contracts, which are homogenous enough to be referred as one type.

Let \( Y \) denote the big contract, or - to be more precise - the big contract’s type. In the example it also had a categorical distribution with a random variable of \( \eta \).

In the general model the distribution of each type can follow any kind of distribution. The Pareto, gamma and lognormal distributions for example
seem to be more realistic from the actuarial point of view, as these are the most common distributions for the numbers of losses.

$N$ describes the set of insurers, which consisted of four elements in the example: $N = \{1, 2, 3, 4\}$.

Insurance companies possess various numbers of contracts of each type of contract. The exact numbers are represented by $\{\alpha_{j}^{i} \cup \{\ast\}\}_{j \in N}$, where the index $j$ represents that of which insurer are we currently speaking of, and $i$ denotes the type of contract. An extension was made regarding the types, as $\ast$ denotes the big contract, and it indicates whether the given insurance company possess it or not.

The premium in this model is exogenous, and the market participants are price takers. In this situation they are competing and cooperating for the higher profit, rather than offering lower premia. In the reinsurance model the premium is denoted by $d$, which was equal to 270 units in the example.

The last, but still very important element of the reinsurance model is the reserve function. How an insurer determines how much she should reserve is defined by a measure of risk, and it will be denoted from now on as $\rho$. In the example we used the value at risk at 99%, but obviously other coherent measures of risk can be used as well.

The previously described data not only give enough information for actuaries to calculate the appropriate premia and expected loss for the insurers and reinsurers, but also provide an opportunity to translate the situation into a cooperative TU-game, the Reinsurance Game. The construction however is not always unambiguous, but it is easy to see, that the previously described model can always be translated into a Reinsurance Game.

To sum up, we present the reinsurance problem in general:

\[
\left\{ I, \{X_{i}\}_{i \in I}, Y, N, \left\{ \alpha_{j}^{i} \cup \{\ast\} \right\}_{j \in N}, d, \rho \right\}
\]

### 3.2 Construction of the Reinsurance Game

The following description is applicable for each and every reinsurance model which is given with the following form:

\[
\left\{ I, \{X_{i}\}_{i \in I}, Y, N, \left\{ \alpha_{j}^{i} \cup \{\ast\} \right\}_{j \in N}, d, \rho \right\}
\]

Let $S$ denote a subset of $N$. For each $S$ coalition we define $\alpha_{S}^{i} \cup \{\ast\}$ as the sum of all $\alpha_{k}^{i} \cup \{\ast\}$, where $k \in S$. 
As a first step we calculate the lowest sufficient reserve for each $S$ coalition without the big contract by using the given measure of risk. This amount was denoted by $R^S_1$.

$$R^S_1 = \rho \left( \sum_{i \in I} \sum_{j \in S} \alpha^i_j \xi^i \right)$$

Next we calculate the lowest sufficient reserve for each $S$ coalition with the big contract. The calculation goes the same way, the only difference is that $Y$ is also taken into consideration. This reserve has been denoted by $R^S_2$.

$$R^S_2 = \rho \left( \eta + \sum_{i \in I} \sum_{j \in S} \alpha^i_j \xi^i \right)$$

The difference between the two situations should be calculated, and it should be subtracted from the given premium, $d$. This way we can determine the profit of each coalition. However, as there is one company that is responsible for organizing the reinsurance (from now on he will be denoted as $i^*$) of the given policyholder, the coalitions which do not contain him should worth 0. At this point we can create a game, as the following:

$$v(S) = \begin{cases} \max \{ d - (R^S_2 - R^S_1), 0 \} & \text{if } i^* \in S \\ 0 & \text{if } i^* \notin S \end{cases}$$

The last step is the most complicated one. We have to make sure that the grand coalition does not contain any unnecessary players, who also might demand more than their marginal contribution to the grand coalition, otherwise the coalition would be unstable.

For the reinsurers we require that their joint contribution to the grand coalition should be greater or equal to the sum of their separate contribution. This requirement is to embrace the synergic effect of the cooperation: if separately they can make a higher contribution than together, it simply means that together they are less effective than separately. This results in the opposite of the synergic effect, therefore it is desirable to avoid this situation. Formally:

$$v(N) - v(S) \geq \sum_{i \in N \setminus S} v(N) - v(N \setminus \{i\}) \text{ if } i^* \in S$$

How we decide to eliminate the players is irrelevant in this case: we only want to make sure that with the remaining reinsurers the cooperation is stable.
3.3 Reinsurance Games and Big Boss Games

Theorem 5. Reinsurance Games are Big Boss Games.

Proof. Big Boss Games require two conditions, as it has previously been presented. It is easy to prove, that the above constructed Reinsurance Game satisfies both of them.

The construction of the Reinsurance Game makes sure that the first condition is valid. Those coalitions, which do not contain the cedant ($i^*$), is the Big Boss of the Reinsurance Game, and without her each group of insurers can only get zero profit. Therefore $v(S) = 0$ if $i^* \notin S$ for all $S \subseteq N$ coalitions. Hence, the first condition is satisfied.

The second condition can also be derived from the construction of the Reinsurance Game. The last step, which was a stability requirement, and which made sure the reinsurers together contribute more to the grand coalition than the sum of their individual contributions, basically defined the second condition of the Big Boss Games. Therefore, if the construction was carefully followed, then the second condition of Big Boss Games will also be satisfied.

□

4 Conclusion

Reinsurance situations can always be modeled with Big Boss Games, however actuaries never do. This construction proves that the game theoretical approach may provide a unique point of view on these situations, therefore further research on this topic might be advisable.

References


