A paired comparisons ranking and Swiss-system chess team tournaments

László Csató*

Department of Operations Research and Actuarial Sciences
Corvinus University of Budapest

October 15, 2012

Abstract

The paper reveals that the Weighted Logarithmic Least Squares Method used for deriving evaluations of alternatives from incomplete pairwise comparison matrices ($iWLLSM$) coincides with the recursive Buchholz ranking method applied in generalized tournaments, despite their different approach and calculation. We study it with respect to a set of new properties and present the strength of the method in the case of Swiss-system chess team tournaments.

Keywords: Multicriteria decision making, Incomplete pairwise comparison matrix, Tournament ranking, Recursive Buchholz, Swiss-sytem tournaments

1 Introduction

In the world of innumerable complex choices, rankings are becoming an important tool for both individuals and organizations in order to make decisions. The paper discusses a specific ranking method based on paired comparisons. They are present in various situations such as sport results, product testings or evaluations of alternatives. There are two different approaches, tournament ranking and pairwise comparison matrices. We will outline them and highlight the connection of some basic concepts of these two fields. The recursive Buchholz method based on a weighted least squares error function is examined from an axiomatic point of view, then it will be applied for a particular sport problem. Some technical difficulties may arise and notations sometimes seem to be complicated, but it is an inevitable consequence of moving on the frontier of two seemingly distant research fields.

The paper is structured as follows. Section 2 deals with (incomplete) pairwise comparison matrices, defines the incomplete weighted version of the Logarithmic Least Squares Method ($iWLLSM$) and shows its calculation, a key issue regarding the proof of some properties. In Section 3 the recursive Buchholz method is introduced, and its common roots with $iWLLSM$ are revealed. Section 4 discusses several known and some entirely new properties of the method and gives an alternative proof for one. Section 5 presents a variant of the procedure specified for Swiss-system chess team tournaments, with respect to currently used rankings and tie-breaking rules. The main results and remaining problems are summarized in Section 6. The paper builds upon [4] and especially upon [14]; readers not familiar with them are recommended to study thoroughly, although all the necessary concepts will be defined here.

2 Generalizations of the Logarithmic Least Squares Method

2.1 Pairwise comparison matrices

Pairwise comparison matrices play an important role in Multicriteria Decision Making problems since Saaty introduced the Analytic hierarchy process (AHP) method [20]. It aims to derive priorities for

*e-mail: laszlo.csato@uni-corvinus.hu

a set of alternatives based on their pairwise comparisons; the main idea is that the evaluation of two opportunities is simpler than to give the weights explicitly. This concept is readily applicable for sport tournaments, where the matches are between two opponents, and the final aim is to rank them according to their results.

Matrix $A = (a_{ij})$ of size $n \times n$ is called pairwise comparison matrix if it is positive and reciprocal, that is, $a_{ij} > 0$ and $a_{ji} = 1/a_{ij}$ for all $i, j = 1, 2, \dots, n$. Because of the reciprocity condition $a_{ii} = 1$ for all $i = 1, 2, \dots, n$. Matrix element a_{ij} is the value assigned by the decision-maker for her preference of alternative i over alternative j (the numerical answer to the question 'How many times alternative i is better than alternative j?'). Analogously, it could reflect the result of a game between two contestants. The key issue is how to transform the known outcomes into pairwise comparison ratios; logical considerations suggest that wins should be represented by values above 1 (thus losses by the reciprocal of it, which is lower than 1) and draws by 1.

Pairwise comparisons serve as a basis for determining the positive weight vector $w = (w_i) \in \mathbb{R}^n_+$, where w_i/w_j somehow approximates the pairwise comparison a_{ij}. It shows that the priority vector w is invariant under multiplication with positive scalars, therefore it should be normalized, often achieved by $\sum_{i=1}^n w_i = 1$. Matrix A is called consistent if $a_{ik} = a_{ij}a_{jk}$ for all $i, j, k = 1, 2, \dots, n$. The rank of a consistent matrix is one and there exists a unique (up to a multiplication by a positive scalar) positive n-dimensional vector w such that $a_{ij} = w_i/w_j$ for all $i, j = 1, 2, \dots, n$. It is a basic requirement that any weight-deriving method yield this vector for consistent matrices.

In the inconsistent case there are some 'errors' in comparisons, consequently, the weights should be estimated, by aggregating the $n \times (n-1)/2$ elements above the diagonal into an n-dimensional vector. Sport tournaments often call forth inconsistent matrices, as circular wins $(a_{ij}, a_{jk}, a_{ki} > 1)$ mean inconsistent (moreover, intransitive) triads. Saaty introduced the Eigenvector Method (EM) for a solution of this problem: the Perron theorem [18] ensures that a positive matrix has a dominant eigenvalue of multiplicity one and an associated strictly positive eigenvector, the latter giving the weights that are looked for [20]. Eigenvectors are invariant under scalar multiplication, similarly to the weight vector, thus normalization is arbitrary.

Other techniques mainly use some kind of distance-minimization. The Least Squares Method (LSM) [6] approximates the matrix elements a_{ij} by minimizing the Euclidean distance. Its main disadvantage is the nonconvexity of the problem, it may have several local or global optima, despite for a class of matrices it has a unique global optimum [11]. The Logarithmic Least Squares Method ($LLSM$) [7, 8, 10] measures the proximity of a_{ij} and its approximation w_i/w_j in a logarithmic sense, the mathematical formula of the problem is:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\log a_{ij} - \log \left(\frac{w_i}{w_j} \right) \right]^2 ,$$
$$w_i > 0, \quad i = 1, 2, \dots, n.$$

The unique solution is given by the geometric means of the rows of matrix A (which analogously should be normalized) [8, Theorem 3]. Logarithmic scale means that the multiplicative definition of the pairwise comparison matrix is transformed into an additive one, used in the tournament approach presented in Section 3.

2.2 Incomplete pairwise comparison matrices

In the usual framework all pairwise comparisons are supposed to be available, which is not the case in some sport tournaments: tight time and infrastructural limits or the physical capabilities of the players can hinder the accomplishment of all possible matchings, especially if the number of participants is large. [13] introduced incomplete pairwise comparison matrices for similar problems, where some elements of the matrix (outside the diagonal) are unknown. In the example below, they are denoted by $*$:

$$A = \begin{pmatrix} 1 & * & a_{13} & a_{14} \\ * & 1 & a_{23} & * \\ 1/a_{13} & 1/a_{23} & 1 & a_{34} \\ 1/a_{14} & * & 1/a_{34} & 1 \end{pmatrix}.$$

In general, there are two ways to derive a weight vector from incomplete pairwise comparison matrices. The first is to represent the d missing elements in the upper triangle by the variables $x = [x_1, x_2, \dots, x_d] \in$

$\mathbb{R}_+$ and regard the new matrix $A(x)$ to be a function of them. Now it is compiled as a mathematical object, thus x could be chosen in some ways, for instance, by minimizing the inconsistency of the matrix $A(x)$. This treatment was used by [4] to extend Saaty's EM to the incomplete case.

For techniques with an explicit objective function there exists another natural way: take only the known elements into it. It reflects the obvious meaning of incomplete matrices that some information is lost due to unknown comparisons, therefore they should not be taken into account. This solution suggests the incompleteness cannot be boundless if all alternatives are intended to be evaluated on a common scale. In order to discuss this issue, the graph interpretation of pairwise comparisons seems to be useful.

(Incomplete) pairwise comparison matrices can be represented by directed graphs [12, 16], but for our purpose the undirected variant is needed. Let $A \in \mathbb{R}^{n \times n}$ be an incomplete pairwise comparison matrix. Then the graph associated with A is $G := (V, E)$, where $V = \{1, 2, \dots, n\}$, the vertices correspond to the alternatives, and $E = \{e(i,j) : a_{ij} \text{ is known and } i \neq j\}$, therefore the set of edges E represents the structure of known elements. It makes possible to formalize the objective function of $LLSM$ for incomplete matrices ($iLLSM$) as [4]:[1]

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \chi\left(e(i,j) \in E\right) \left[\log a_{ij} - \log \left(\frac{w_i}{w_j} \right) \right]^2,$$

$$\sum_{i=1}^{n} w_i = 1,$$

$$w_i > 0, \quad i = 1, 2, \dots, n,$$

where $\chi\left(e(i,j) \in E\right)$ is the characteristic function of $e(i,j) \in E$:

$$\chi\left(e(i,j) \in E\right) = \begin{cases} 1 & \text{if } e(i,j) \in E, \\ 0 & \text{if } e(i,j) \notin E. \end{cases}$$

Theorem 2.1. *The solution of the incomplete logarithmic least squares problem (iLLSM) is unique if and only if graph G corresponding to the incomplete pairwise comparison matrix A is connected.*

Proof. See [4]. □

2.3 The incomplete Weighted Logarithmic Least Squares Method

In some cases it seems to be a natural assumption that the importance or reliability of pairwise comparisons are not the same. For this purpose, the objective function of $iLLSM$ is extended by account for nonnegative weights[2] m_{ij} of the comparison between alternatives i and j ($iWLLSM$):

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \left[\log a_{ij} - \log \left(\frac{w_i}{w_j} \right) \right]^2,$$

$$\sum_{i=1}^{n} w_i = 1,$$

$$w_i > 0, \quad i = 1, 2, \dots, n.$$

As the weights refer to the pairs (i, j), $m_{ij} = m_{ji}$ is assumed for all $i, j = 1, 2, \dots, n$, thus they are symmetric. Moreover, $m_{ij} = 0$ if and only if alternatives i and j are not compared: $m_{ij} = 0 \Leftrightarrow e(i,j) \notin E$.[3] It means that G becomes an undirected, but weighted graph. It is an extension of the original problem by the generalization of the objective function.

[1] Instead of the missing elements a_{ij} arbitrary positive numbers can be written, because their weights in the objective function are 0.

[2] In the literature of pairwise comparison matrices, weights usually refer to the final evaluations of the alternatives (to the vector w), not to the importance of comparisons. We hope that the double use of the term will not confuse the readers and the context makes clear its meaning.

[3] It seems to be a natural, but restrictive assumption. However, it is an unfair decision to give a weight $m_{ij} = 0$ if there is some available information about the pairwise relation of i and j regardless to its possible high uncertainty or unreliability. Analogous argument can be applied to exclude the case $m_{ij} < 0$.

Theorem 2.2. *The solution of the incomplete weighted logarithmic least squares problem (iWLLSM) is unique if and only if graph G corresponding to the incomplete pairwise comparison matrix A is connected.*

Proof. The proof is given by [3] and is based on the Laplacian matrix of the weighted graph G as follows. By introducing the notations

$$r_{ij} = \log a_{ij},$$

$$y_i = \log w_i$$

for all $i, j = 1, 2, \ldots, n$, an unconstrained ($y_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$) optimization problem is defined. The first-order conditions of the optimality give a system of linear equations:

$$\begin{pmatrix} d_1 & -m_{12} & 0 & \ldots & -m_{1n-1} & 0 \\ -m_{12} & d_2 & -m_{23} & \ldots & 0 & -m_{2n} \\ 0 & -m_{23} & d_3 & \ldots & 0 & -m_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -m_{1n-1} & 0 & 0 & \ldots & d_{n-1} & 0 \\ 0 & -m_{2n} & -m_{3n} & \ldots & 0 & d_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_{n-1} \\ R_n \end{pmatrix},$$

where d_i denotes the sum of weights of the edges incident to node i, $d_i = \sum_{j=1, j\neq i}^{n} m_{ij}$ (in the unweighted case d_i is the degree of node i). On the right-hand side $R_i = \sum_{i, m_{ij}>0} r_{ij}$, which is the logarithm of the product of known elements in the ith row. The above equation is denoted by $L(G)y = R$, where $L(G)$ is the Laplacian matrix of the weighted graph G. In the objective function only the ratios of weights count, therefore some kind of normalization is needed, but should not cling to $\sum_{i=1}^{n} y_i = 0$.

The solution can be computed in an analogous way to the unweighted case of $iLLSM$ by [4]. Normalize the weight vector as $y_n = 0$ and take the upper-left $(n-1)\times(n-1)$ submatrix of the Laplacian matrix L(G) denoted by $L_{-1}(G)$, which has an inverse. Similarly, let the corresponding first $n-1$ components of R and y be R_{-1} and y_{-1}, respectively. Then $y_{-1} = [L_{-1}(G)]^{-1} R_{-1}$ or $y_{-1} = \Lambda R_{-1}$ for short, where $\Lambda = [L_{-1}(G)]^{-1}$ and $y_n = 0$. □

This extension of $LLSM$ will be referred to as the incomplete Weighted Logarithmic Least Squares Method ($iWLLSM$).

Remark 1. *The uniqueness of the solution depends only on the positions of matrix elements with positive weights $m_{ij} > 0$ (the structure of the graph G), and does not depend on the values of comparisons.*

3 Tournaments and the recursive Buchholz method

In this section we mainly use the concepts of [14]. A tournament is a pair (N, T), where $N = \{1, 2, \ldots, n\}$ is the finite set of players and $T \in \mathbb{R}^{n\times n}$ is a tournament matrix. $T_{ij} \geq 0$ represents the aggregate score of player i against player j, and $T_{ii} = 0$ for all $i \in N$. Each tournament defines a matches matrix $M(T) = T + T^\top$, where M_{ij} is the number of matches between players i and j. The total number of matches played by i is $m_i = \sum_{j\in N} M_{ij}$, that is, $m = Me$. The proportion of player i's matches against player j is denoted by $\bar{M}_{ij} = M_{ij}/m_i$. A tournament is called round-robin if $M_{ij} = 1$ for all $i, j \in N; i \neq j$; a round-robin tournament is 'complete' in which each player has played exactly once against any other player. In the pairwise comparison approach it means the standard case when the matrix is complete and all weights in the objective function are equal to 1. It is assumed that all players are compared directly or indirectly, through other players. In the graph representation of the tournament, where the nodes are the players and the edges are the matches between them, it coincides with the connectedness of the graph.

A ranking method ϕ assigns to each tournament (N, T) a weak order $\phi(T)$ on N, which is transitive and complete. For a tournament (N, T), a vector $q \in \mathbb{R}^n$ is a rating vector, where q_i is a measure of the performance of player i in the tournament. A ranking method is induced by a rating vector: for each ranking method ϕ there is an underlying rating vector q such that the players are ranked according to it, i.e., ϕ ranks i weakly above j if and only if $q_i \geq q_j$.

Definition 3.1. *The* score *rating vector s is the vector of average scores: $s_i = \sum_{j\in N} T_{ij}/m_i$.*

Definition 3.2. *The* Buchholz *rating vector* q^b *is given by* $q^b = \bar{M}s + s$.

Definition 3.3. *The* recursive Buchholz *rating vector* x *is the unique solution of the system of linear equations* $x^\top e = 0$ *and* $\bar{M}x + \hat{s} = x$, $\hat{s} = s - 0.5e$.[4]

It takes account of not only the score (s_i) and the average score of the opponents ($\bar{M}s_i$) as Buchholz, but gives an infinite depth to this argument. This method has strong links to $iWLLSM$.

Theorem 3.1. *The recursive Buchholz rating vector* x *provides the same ranking as the solution of a correspondingly defined weighted least squared errors problem:*

$$\min_{r \in \mathbb{R}^n} \sum_{i,j \in N} M_{ij} \left[\frac{T_{ij} - T_{ji}}{M_{ij}} - (q_i - q_j) \right]^2 .$$

Proof. See [14]. □

It is the same objective function as in the case of $iWLLSM$ by taking $M_{ij} = m_{ij}$ and $(T_{ij} - T_{ji})/M_{ij} = (T_{ij} - T_{ji})/m_{ij} = r_{ij}$. As [14] mentions, the recursive Buchholz rating vector x does not actually minimize the above objective function, but $2x$ does.[5] Hence in the followings, we will call the latter vector to recursive Buchholz.

Despite the same result, their calculation is different; for the $iWLLSM$, the system of linear equations is given by $L(G)y = R$, and for the recursive Buchholz, it is $(I - \bar{M})x = 2\hat{s}$, while the normalizations $y^\top e = 0$ and $x^\top e = 0$ are the same. In short, the recursive Buchholz uses an 'average' sense and the $iWLLSM$ applies a direct approach: regarding the former, $\hat{s}$ is an average and $\bar{M}$ has a proportional meaning, while the latter is based on sums and graph theory concepts. They differ significantly if the vector m has some variance, and at first sight the equivalence of the two systems of linear equations is rather not obvious. Recursive Buchholz seems to be a bit more straightforward, but it has a price since the properties of the Laplacian matrix cannot be exploited.[6]

Example 3.1. *The different calculation of recursive Buchholz and* $iWLLSM$*:*

$$T = \begin{pmatrix} 0 & 0 & 3/4 & 1 \\ 0 & 0 & 3/4 & 0 \\ 1/4 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1/4 & 0 \end{pmatrix} \text{ and } \log(A) = \begin{pmatrix} 0 & * & 1/2 & 1 \\ * & 0 & 1/2 & * \\ -1/2 & -1/2 & 0 & 1/2 \\ -1 & * & -1/2 & 0 \end{pmatrix}; M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix};$$

$$\bar{M} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}, \ I - \bar{M} = \begin{pmatrix} 1 & 0 & -1/2 & -1/2 \\ 0 & 1 & -1 & 0 \\ -1/3 & -1/3 & 1 & -1/3 \\ -1/2 & 0 & -1/2 & 1 \end{pmatrix} \text{ and } \hat{s} = \begin{pmatrix} 0.3750 \\ 0.2500 \\ -0.0833 \\ -0.3750 \end{pmatrix};$$

$$L = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1.5 \\ 0.5 \\ -0.5 \\ -1.5 \end{pmatrix}; \text{ therefore the solution is } y = 2x = \begin{pmatrix} 0.375 \\ 0.375 \\ -0.125 \\ -0.625 \end{pmatrix}.$$

It is worth to emphasize two differences between the $iWLLSM$ and recursive Buchholz methods. The first is that the pairwise comparison matrix contains fewer information than the tournament matrix; it should be supplemented with the matches matrix, which gives the weights in the objective function. The other is about the necessary conditions of a unique solution. [14] assumes the tournament matrix to be irreducible, but in fact only the connectedness of graph G is required. That is more general since it demands that for every pair of players $i, j \in N, i \neq j$, there exists a sequence (chain) of players $i = k_0, k_1, \ldots, k_{n-1}, k_n = j$, such that k_ℓ has played against $k_{\ell+1}$ for each $\ell \in \{0, 1, \ldots, n-1\}$.[7] This ranking method will be referred to as recursive Buchholz in order to highlight its origin.

[4] The uniqueness is guaranteed by the assumed connectedness of the tournament, since recursive Buchholz is equivalent to $iWLLSM$.

[5] It refers to the property scale-invariance (SI), discussed in Section 4.

[6] This can be partly responsible for the complicated proof of the property nonnegative responsiveness to the beating relation ($NNRB$) by [14], discussed in the next section.

[7] Irreducibility requires that in the sequence all k_ℓ have scored against $k_{\ell+1}$. [14] builds recursive Buchholz upon the recursive performance ranking method defined by [5]. Here it is assumed that the matches matrix M is irreducible and not block anti-diagonal, that is, do not exist two disjoint subsets of players such that the players in each subset have played only against the players of the other subset.

4 Some properties of the recursive Buchholz method

4.1 Scale-invariance and acyclicity

[14] discusses fourteen basic features of ranking methods for generalized tournaments. In this section, they will be examined only if they have some signifance for our topic. We will follow [14] to avoid the difficulties arising from different notations.

The tournament will be represented by a results matrix Θ (it is the same as $\log(A)$ for a properly defined pairwise comparison matrix), where $\Theta_{ij} = (T_{ij} - T_{ji})/M_{ij}$, thus $\Theta_{ji} = -\Theta_{ij}$, and the numbers of matches between players by a matches matrix M (consequently, M_{ij} is the weight of the pairwise comparison of players i and j in the objective function to be minimized). With a slight abuse of notation, a tournament (N, Θ, M) will be referred to as (Θ, M) if it does not cause inconvenience. It is not a parsimonous notation, but it will be useful since the results matrix Θ solely determines R and the matches matrix M solely determines $L(G)$.

The recursive Buchholz rating vector y can be calculated as $y_{-1} = \Lambda R_{-1}$, where $\Lambda = [L_{-1}(G)]^{-1}$ is the inverse of the upper-left $(n-1) \times (n-1)$ submatrix of the Laplacian matrix $L(G)$ of the weighted graph G corresponding to the matches matrix M and $y_n = 0$. Some basic properties are introduced in order to outline the main features of this ranking method.

Definition 4.1. ***Anonymity*** *(ANO): Let $i, j \in N$ and let (Θ', M') be the tournament obtained from (Θ, M) by permuting columns i and j and rows i and j. ϕ satisfies ANO if the rankings $\phi(\Theta, M)$ and $\phi(\Theta', M')$ are the same but with players i and j interchanged.*

ANO is a standard property of ranking methods, it just requires that the rankings should be independent of the player's 'names'. Recursive Buchholz does not take into account this information, it is anonymous.

Definition 4.2. ***Independence of irrelevant matches*** *(IIM): Take four different players $i, j, k, \ell \in N$. Suppose that (Θ, M) and (Θ', M') are identical, except for the results and number of matches between k and ℓ. ϕ satisfies IIM if the relative ranking of i and j in both (Θ, M) and (Θ', M') are the same: $y_i(\Theta, M) \geq y_j(\Theta, M)$ if and only if $y_i(\Theta', M') \geq y_j(\Theta', M')$.*

[14] demonstrates through an example that recursive Buchholz violates *IIM*. It is easy to see that the score method satisfies *IIM*. While independence of irrelevant comparisons is an acceptable property for round-robin tournaments, when players have different opponents (or results have different intensities), in the incomplete case precisely those 'irrelevant' matches played among other teams reflect the strength of opponents.

The following two properties are not defined by [14], but they are straightforward from our setting of tournaments in the form Θ, M.

Definition 4.3. ***Scale invariance*** *(SI): Let $(\Theta' = \alpha\Theta, M)$ be the tournament obtained from (Θ, M) by multiplying the results matrix Θ with a constant $\alpha > 0, \alpha \in \mathbb{R}^n$. ϕ satisfies SI if $\phi(\Theta', M) = \phi(\Theta, M)$.*

It means that the ranking is not sensitive to the choice of the measure of results; for example, if wins are represented by an arbitrary number $\kappa > 0$, draws by 0 (in the results matrix Θ unplayed matches are also represented by 0) and losses by $-\kappa$, then the recursive Buchholz ranking is unique.

SI is different from homogeneity (*HOM*) defined by [14], which requires that $\phi(\alpha T) = \phi(T)$ for all $\alpha > 0$. Since T and kT results in (Θ, M) and $(\Theta, \alpha M)$, respectively, it is not the same as *SI*, but *HOM* can be regarded as a 'dual' version of *SI*.

Proposition 4.1. *Recursive Buchholz satisfies SI.*

Proof. It is clear that $L(G)$ is not affected by the choice of α. The results matrix Θ induces the equation system $y_{-1} = [L_{-1}(G)]^{-1} R_{-1}$, which becomes to $y'_{-1} = [L_{-1}(G)]^{-1} \alpha R_{-1}$ for $\Theta' = \alpha\Theta$, consequently, $y' = \alpha y$ implying the ranking is unchanged. □

Definition 4.4. ***Acyclicity*** *(AC): Let (Θ, M) be a tournament such that $\Theta_{ij} = \Theta_{jk} = \Theta_{ki}$ for three different players $i, j, k \in N$. Let (Θ', M) be a tournament identical to (Θ, M), except that $\Theta'_{ij} = -\Theta_{ij}, \Theta'_{jk} = -\Theta_{jk}$ and $\Theta'_{ki} = -\Theta_{ki}$. ϕ satisfies AC if $\phi(\Theta', M) = \phi(\Theta, M)$.*[8]

[8]Note that $M_{ij} = M_{jk} = M_{ik}$ is not required, the number of matches among the three players can be arbitrary.

AC means insensitivity to the direction of circular wins. If the intensities are eliminated ($\Theta_{ij} = \Theta_{jk} = \Theta_{ki}$), and it is found that i beats j, j beats k and k beats i, then the final result will not change if the direction is turned up: j beats i, i beats k and k beats j. It has greater significance if the intensities are limited, for instance by choosing a *binary* representation of wins, draws and losses similar to the example for SI and to generalized $TB1$ in Proposition 5.1.

Proposition 4.2. *Recursive Buchholz satisfies AC.*

Proof. R_i, R_j and R_k are not influenced by the modification, because $\Theta'_{ij} + \Theta'_{ik} = -\Theta_{ij} - \Theta_{ik} = -\Theta_{ij} + \Theta_{ki} = 0$, and so on. The Laplacian matrix $L(G)$ does not change. □

4.2 The connection of bridge player independence and positive responsiveness to the beating relation

Let (N, θ, M) be a tournament. A player $b \in N$ is called a *bridge player* in (N, θ, M) if there exist sets $N^1, N^2 \subseteq N$ such that $|N^1| \geq 2$, $|N^2| \geq 2$, $N^1 \cup N^2 = N$, $N^1 \cap N^2 = \{b\}$ and $M_{ij} = 0$ for all $i \in N^1 \setminus \{b\}, j \in N^2 \setminus \{b\}$. It implies that the connectedness of the graph associated with the tournament exclusively depends on b: by removing her, the uniqueness of the solution is lost. In this case two subtournaments (N^1, Θ^1, M^1) and (N^2, Θ^2, M^2) could be defined by taking only the matches among players in N^1 and N^2, respectively.

Definition 4.5. ***Bridge player independence*** *(BPI): Let b be a bridge player in the tournament (N, Θ, M) with the corresponding subtournaments (N^1, Θ^1, M^1) and (N^2, Θ^2, M^2). ϕ satisfies BPI if for all $i, j \in N^1$, $\phi(\Theta, M)$ ranks i weakly above j if and only if $\phi(\Theta^1, M^1)$ ranks i weakly above j, that is, $y(\Theta, M)_i \geq y(\Theta, M)_j \Leftrightarrow y(\Theta^1, M^1)_i \geq y(\Theta^1, M^1)_j$.*

The main idea is that a bridge player cannot manipulate the final ranking in one (sub)tournament by finding some 'friends' in another tournament, and playing with them some arbitrary matches. The connection of the two subtournaments could be made more explicit in certain cases. Let (N, Θ, M) be a tournament. An $M_{ij} > 0$ element of the matches matrix M is called a *critical match* if there exists a partition $N^1 \cup N^2 = N, N^1 \cap N^2 = \emptyset$ such that $i \in N^1$, $j \in N^2$ and $M_{k\ell} = 0$ for all $k \in N^1$ and $\ell \in N^2$ except for $k = i$ and $\ell = j$. It means the graph representing the structure of matrix M is 'weakly' connected, that is, by ignoring the comparison between i and j it becomes unconnected. If the partition N^1 and N^2 satisfies the conditions $|N^1| \geq 2$ and $|N^2| \geq 2$, then i and j are adjacent bridge players, but it remains possible that $|N^1| = 1$ or $|N^2| = 1$. Note that the concepts of bridge player and critical match 'dually' unveil the 'weak' connectedness of the graph: by removing a bridge player (a node) or a critical match (an edge), the resulting subgraph becomes unconnected. The next property establishes the consequences of the existence of a critical match, in terms of the rating vector.

Definition 4.6. ***Critical match preservation*** *(CMP): Let $M_{ij} > 0$ a critical match in the tournament (N, Θ, M) with a partition $N^1 \cup N^2 = N, N^1 \cap N^2 = \emptyset$. y satisfies CMP if $y(\Theta, M)_i - y(\Theta, M)_j = \Theta_{ij}$.*[9]

CMP implies that the difference of weights $y(\Theta, M)_i$ and $y(\Theta, M)_j$ (which is independent of the normalization of $y(\Theta, M)$) is determined exclusively by the result of matches between them, notably Θ_{ij}.

Proposition 4.3. *Recursive Buchholz satisfies CMP.*

Proof. Graphs G^1 and G^2 defined by subtournaments (N^1, Θ^1, M^1) and (N^2, Θ^2, M^2), respectively, have well-defined *internal* rating vectors, that is, the relative weights $y(\Theta^1, M^1)$ and $y(\Theta^2, M^2)$ of players in N^1 and N^2, respectively, are unique and independent from each other due to BPI for $|N^1| \geq 2$ and $|N^2| \geq 2$ and simple intuition for $|N^1| = 1$ or $|N^2| = 1$.[10] Regarding the whole tournament (N, Θ, M), the entire rating vector $y(\Theta, M)$ is determined by $y(\Theta^1, M^1)$ and $y(\Theta^2, M^2)$ minimizing the objective function $\sum_{k,\ell \in N} M_{k\ell} \left[\Theta_{k\ell} - (q_k - q_\ell)\right]^2$, implying that $y(\Theta, M)_i - y(\Theta, M)_j = \Theta_{ij}$ as it is the only difference that can change. □

[9] Note that a weaker version of CMP can be formulated for the ranking ϕ by requiring that ϕ ranks i weakly above j if $\Theta_{ij} \geq 0$.

[10] For tournaments with only one player the rating vector, invariant to multiplication with positive scalars, is trivial.

The following set of properties (supplemented with BPI) are about the monotonicity of a ranking method.

Definition 4.7. ***Positive responsiveness to the beating relation*** *(PRB): Let (Θ, M) be a tournament such that $\phi(\Theta, M)$ ranks player i weakly above j, that is, $y(\Theta, M)_i \geq y(\Theta, M)_j$. Let (Θ', M) be a tournament identical to (Θ, M), except that there is $k \in N \setminus \{i\}$ such that $M'_{ik} = M_{ik}$ and $\Theta'_{ik} > \Theta_{ik}$ (implying $\Theta'_{ki} = -\Theta'_{ik} < -\Theta_{ik} = \Theta_{ki}$). ϕ satisfies PRB if $\phi(\Theta', M)$ ranks i strictly above j, that is, $y(\Theta', M)_i \geq y(\Theta', M)_j$.*[11]

Since recursive Buchholz satisfies BPI [14], it is clear that it violates positive responsiveness to the beating relation (PRB), because a better performance of a bridge player against players in another subtournament has no effect on her relative ratings against players in the chosen subtournament.

In this view it is useful to relax the strict inequality in PRB, resulting in nonnegative responsiveness to the beating relation ($NNRB$). Recursive Buchholz satisfies $NNRB$ as it has been proven by [14]. However, our solution seems to be a bit simpler and has some interesting consequences.

Definition 4.8. ***Nonnegative responsiveness to the beating relation*** *($NNRB$): Let (Θ, M) be a tournament such that $\phi(\Theta, M)$ ranks player i weakly above j, that is, $y(\Theta, M)_i \geq y(\Theta, M)_j$. Let (Θ', M) be a tournament identical to (Θ, M), except that there is $k \in N \setminus \{i\}$ such that $M'_{ik} = M_{ik}$ and $\Theta'_{ik} > \Theta_{ik}$ (implying $\Theta'_{ki} = -\Theta'_{ik} < -\Theta_{ik} = \Theta_{ki}$). ϕ satisfies $NNRB$ if $\phi(\Theta', M)$ ranks i weakly above j, that is, $y(\Theta', M)_i \geq y(\Theta', M)_j$.*

It means that player i cannot lose if she performs better, or, regarding alternatively, no player could manipulate the method by unilaterally performing weaker.

Theorem 4.1. *Recursive Buchholz satisfies $NNRB$.*

Proof. Since the nth player could be chosen arbitrarily, the weights can be normalized for both tournaments by $y_k = y'_k = 0$, thus in the proof $k = n$ can be assumed without loss of generality. It is enough to show that $y'_i - y_i \geq y'_j - y_j$ for all $j \in N \setminus \{i, n\}$ and $y'_i - y_i \geq 0$ as the Laplacian matrix $L(G)$ is fixed. In the equation system $y'_{-1} = \Lambda R'_{-1}$ the matrix Λ is unchanged, but $R'_i > R_i$ as $\Theta'_{ik} > \Theta_{ik}$ and $R'_j = R_j$ for all $j = N \setminus \{i, n\}$ (the last equation belonging to n is eliminated). We prove that matrix Λ is nonnegative and $\Lambda_{ii} \geq \Lambda_{ji}$ for all $i, j = 1, 2, \dots, n-1$.

Assume that some elements in the ith column are negative and let $\Lambda_{ji} = \min\{\Lambda_{ki} : 1 \leq k \leq n-1\}$ be (one of) the minimal element of the ith column. The equation $L_{-1}(G)\Lambda = I$, where I is the unit matrix, gives $d_j \Lambda_{ji} - \sum_{k=1, k\neq j}^{n-1} M_{jk}\Lambda_{ki} \leq \left(d_j - \sum_{k=1,k\neq j}^{n-1} M_{jk}\right)\Lambda_{ji} \leq 0$ for the jth row, since $\Lambda_{ji} \leq \Lambda_{ki}$, $\Lambda_{ji} < 0$, $M_{jk} \geq 0$ and $d_j \geq \sum_{k=1,k\neq j}^{n} M_{jk}$. The properties of inverse imply the sum is exactly 0 (if $j \neq i$) or 1 (if $j = i$), which means $j \neq i$. The equality holds if and only if $d_j = \sum_{k=1,k\neq j}^{n-1} M_{jk}$ and $\Lambda_{ji} = \Lambda_{ki}$ for all k with $M_{jk} > 0$. Therefore in the deduction all nodes k adjacent to j (the players who has played against j) can substitute j. It implies the deduction is true for all nodes connected to j, and the connectedness of graph G ensures that there exists $k \in N$ such that $M_{kn} > 0$, which is a contradiction. Hence Λ is a nonnegative matrix and $y'_i - y_i = \Lambda_{ii}(R'_i - R_i) \geq 0$.

Now we know that $\Lambda_{ji} \geq 0$ for all $i, j = 1, 2, \dots, n-1$. Let $\Lambda_{ji} = \max\{\Lambda_{ki} : 1 \leq k \leq n-1\}$ be (one of) the maximal element in the ith column. Assume that $\Lambda_{ji} > \Lambda_{ii}$, thus $y'_j - y_j = \Lambda_{ji}(R'_i - R_i) > \Lambda_{ii}(R'_i - R_i) = y'_i - y_i$. $L_{-1}(G)\Lambda = I$ gives $d_j \Lambda_{ji} - \sum_{k=1,k\neq j}^{n-1} M_{jk}\Lambda_{ki} \geq \left(d_j - \sum_{k=1,k\neq j}^{n-1} M_{jk}\right)\Lambda_{ji} \geq 0$ for the jth row. The properties of inverse imply the sum is exactly 0. The equality holds if and only if $d_j = \sum_{k=1,k\neq j}^{n} M_{jk}$ and $\Lambda_{ji} = \Lambda_{ki}$ for all k with $M_{jk} > 0$. Thus k can substitute j in the deduction. Repeat it to conclude that $\Lambda_{ji} = \Lambda_{ii}$ because i and j are connected, which contradicts with the assumption of $\Lambda_{ji} > \Lambda_{ii}$. □

Remark 2. *It is recommended to study the proof for $NNRB$ further and examine why PRB is violated. It requires that $\Lambda_{ji} < \Lambda_{ii}$ for all $j \in N \setminus \{i, n\}$ (if it holds, $\Lambda_{ii} > 0$ is implied because Λ is a nonnegative matrix), that is, the assumption is now $\Lambda_{ji} \geq \Lambda_{ii}$. Similarly, from the equation system $L(G)y = R$ for the jth row $d_j \Lambda_{ji} - \sum_{k=1,k\neq j}^{n-1} M_{jk}\Lambda_{ki} \geq \left(d_j - \sum_{k=1,k\neq j}^{n-1} M_{jk}\right)\Lambda_{ji} \geq 0$, but it should be exactly 0.*

[11] Note that this should hold in particular for $k = j$.

The equality holds if and only if $d_j = \sum_{k=1,k\neq j}^{n} M_{jk}$ and $\Lambda_{ji} = \Lambda_{ki}$ for all k with $M_{jk} > 0$. The first condition implies that $M_{jn} = 0$ and the second means that all k having $M_{jk} > 0$ can substitute j. Apply the deduction for this k to get that $M_{kn} = 0$ for all k with $M_{jk} > 0$, therefore no nodes adjacent to j is connected with n. Repeat the argument to conclude that there exists a group of players N^1, such that $M_{jn} = 0$ holds for for all $j \in N^1$. Attach i to N^1 and define N^2 as $N^2 = N \setminus N^1$, that is, $i \in N^2$ (and $n \in N^2$, too). Now $i \in N^1 \cap N^2$ is a bridge player in the tournament (N, Θ, M).

It clearly shows the conflict of BPI and PRB for recursive Buchholz. PRB is not true because bridge players cannot gain from better performance against players in the other subtournament relative to players in the chosen subtournament. However, in the lack of bridge players, or for comparisons in the same subtournament, PRB is satisfied because the assumption of Remark 2 leads to contradiction.

Example 4.1. *PRB is not satisfied by recursive Buchholz:*[12]

$$\Theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Theta' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & -0.5 & 0 \end{pmatrix}; M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The rating vectors are

$$\phi(\Theta, M) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \phi(\Theta', M) = \begin{pmatrix} 0.1667 \\ 0.1667 \\ -0.3333 \end{pmatrix}.$$

In this case player 2 beats player 3 instead of the original draw, but it has no effect on its relative strength to player 1. It is the consequence of her key role in the tournament; if she quits, the result cannot be calculated since players 1 and 3 are not compared. The tournament matrix has a special structure, which is not by chance, as Remark 2 shows.

The deductions above offer a way to prove BPI based on the structure of matrix $\Lambda = [L_{-1}(G)]^{-1}$.

Proposition 4.4. *Recursive Buchholz satisfies BPI.*

Proof. Let $i, j \in N^1$ be two players in a chosen subtournament. It could be assumed that $n \in N^2 \setminus \{b\}$ since the latter is not an empty set. It is enough to prove that $\Lambda_{i\ell} = \Lambda_{j\ell}$ for all players $\ell \in N^2 \setminus \{n\}$ when all changes in R_ℓ (including R_b) have the similar consequences for y_i and y_j.

Assume that $\Lambda_{i\ell} > \Lambda_{j\ell}$. Let $\Lambda_{i\ell} = \max\{\Lambda_{k\ell} : k \in N^1\}$ and $\Lambda_{j\ell} = \min\{\Lambda_{k\ell} : k \in N^1\}$ without a loss of generality. If $\Lambda_{b\ell} \geq \Lambda_{i\ell}$, then $j \neq b$, $j \neq \ell$ and $L_{-1}(G)\Lambda = I$ gives $d_j \Lambda_{j\ell} - \sum_{k=1,k\neq j}^{n-1} M_{jk} \Lambda_{k\ell} \leq \left(d_j - \sum_{k=1,k\neq j}^{n-1} M_{jk} \right) \Lambda_{j\ell} = 0$ for the product of the jth row and the ℓth column. $j \neq b$ implies that $d_j = \sum_{k=1,k\neq j}^{n-1} M_{jk}$ since $M_{jn} = 0$. It means the equality holds if and only if $\Lambda_{j\ell} = \Lambda_{k\ell}$ for all k with $M_{jk} > 0$. Thus k can substitute j in the deduction. Repeat it to conclude that $\Lambda_{j\ell} = \Lambda_{i\ell}$ because i and j are connected, which contradicts with the original assumption.[13]

The case $\Lambda_{b\ell} < \Lambda_{j\ell}$ (consequently, $i \neq b$ and $i \neq \ell$) could be proven by a similar argument for the product of the ith row and the ℓth column resulting in $d_i \Lambda_{i\ell} - \sum_{k=1,k\neq j}^{n-1} M_{ik} \Lambda_{k\ell} \geq \left(d_j - \sum_{k=1,k\neq j}^{n-1} M_{jk} \right) \Lambda_{i\ell} = 0$. Since $i, b \in N^1$ are connected, the conclusion is $\Lambda_{i\ell} = \Lambda_{b\ell}$, which is impossible. Hence $\Lambda_{i\ell} = \Lambda_{j\ell}$ for all $\ell \in N^2 \setminus \{n\}$ and $i, j \in N^1$, thus results outside the subtournament (N^1, Θ^1, M^1), able to modify only R_ℓ, affect the ratings y_i and y_j equally. It exactly means that $\phi(\Theta, M)$ ranks i weakly above j if and only if $\phi(\Theta^1, M^1)$ ranks i weakly above j. □

It is worth to make explicit the consequences of Remark 2 and refine $NNRB$.

Definition 4.9. ***Weak positive responsiveness to the beating relation*** *($WPRB$): Let (Θ, M) be a tournament such that $\phi(\Theta, M)$ ranks player i weakly above j, that is, $y(\Theta, M)_i \geq y(\Theta, M)_j$. Let (Θ', M) be a tournament identical to (Θ, M), except that there is $k \in N \setminus \{i\}$ such that $M'_{ik} = M_{ik}$ and $\Theta'_{ik} > \Theta_{ik}$ (implying $\Theta'_{ki} = -\Theta'_{ik} < -\Theta_{ik} = \Theta_{ki}$). ϕ satisfies $WPRB$ if $\phi(\Theta', M)$ ranks i strictly above j ($y(\Theta', M)_i > y(\Theta', M)_j$) if i is not a bridge player or j and k are in a common subtournament, that*

[12] The example reflects the strength of CMP: since there are only critical matches, the rating vector could be deduced from this axiom.

[13] This substitution process stops for the bridge player b because then $M_{bn} = 0$ is not certainly true.

is, they are connected not only through bridge player i. Otherwise, $\phi(\Theta', M)$ should rank i weakly above j, ($y(\Theta', M)_i \geq y(\Theta', M)_j$) if i is a bridge player and players j and k are in different subtournaments, that is, they are connected only through i.

Note that $WPRB$ implies $NNRB$.

Theorem 4.2. *Recursive Buchholz satisfies $WPRB$.*

Proof. It is the consequence of Theorem 4.1, Remark 2 and Proposition 4.4. □

4.3 Homogeneous treatment of victories and self-consistent monotonicity

The following properties connect recursive Buchholz to the score ranking method.

Definition 4.10. ***Score consistency*** *(SCC): ϕ coincides with the score ranking method on the class of round-robin tournaments.*

Consequently, if a ranking method satisfies SCC, then it satisfies IIM on the class of round-robin tournaments. The following property is a generalization of SCC.

Definition 4.11. ***Homogeneous treatment of victories*** *(HTV): Let $i, j \in N$ and (Θ, M) be a tournament such that $M_{ik} = M_{jk}$ for all $k \in N \setminus \{i, j\}$. ϕ satisfies HTV if $\phi(\Theta, M)$ ranks i weakly above j if and only if s_i is not lower than s_j, that is, $y(\Theta, M)_i \geq y(\Theta, M)_j \Leftrightarrow s(\Theta, M)_i \geq s(\Theta, M)_j$.*

AC and HTV clearly have some common features, as both of them means the separation of results (which, for instance, are measured by scores) and the opponents, in the case of recursive Buchholz by the independence of the Laplacian matrix $L(G)$ determined by the matches matrix M and the original evaluations R determined by the results matrix Θ. It suggests that their relation could be made explicit; however, HTV deals with two players and AC with three. It is an open question whether they can be incorporated into a common framework.

[14] proves that recursive Buchholz satisfies HTV, hence SCC. However, we will give an alternative argument, which uses the properties of the Laplacian matrix $L(G)$.

Proposition 4.5. *Recursive Buchholz satisfies HTV.*

Proof. The tournament (Θ, M) induces the equation system $y_{-1} = \Lambda R_{-1}$. Since the order of the players is arbitrary, it is enough to consider the alternatives $i \neq n$ and $j \neq n$. It will be shown that $\Lambda_{ij} = \Lambda_{ji}$, $\Lambda_{ii} = \Lambda_{jj}$ and $\Lambda_{ik} = \Lambda_{jk}$ for all $k \in N \setminus \{i, j, n\}$. As $y_i = \sum_{k=1}^{n} \Lambda_{ik} R_k$ and $y_j = \sum_{k=1}^{n} \Lambda_{jk} R_k$, $y_i - y_j = \Lambda_{ii} R_i + \Lambda_{ij} R_j - \Lambda_{jj} R_j + \Lambda_{ji} R_j = (\Lambda_{ii} - \Lambda_{ij})(R_i - R_j)$. In the proof of $NNRB$ it was shown that $\Lambda_{ii} - \Lambda_{ij} \geq 0$, thus $y_i - y_j \geq 0 \Leftrightarrow s_i - s_j \geq 0$. □

The main weakness of recursive Buchholz is that it does not satisfy self-consistent monotonicity (SCM) as the consequence of its additive feature [14]. The formal definition of SCM is rather complicated, but it has a nice intuitive interpretation. Ratings from incomplete tournaments could be used to gain a forecast for unknown elements in the form of $\phi_i - \phi_j$. Thus a 'predicted' tournament can be made by ignoring some comparisons (assuming that the associated graph remains connected) and writing the predicted results instead of the ignored elements. The optimal values of the objective function for the original and 'predicted' tournaments are the same if and only if the predicted result coincides with the ignored one. The problem is that the forecast $y_i - y_j$ for Θ_{ij} can be larger than its maximal value 1 (since $T_{ij} \leq M_{ij}$ for all $i, j \in N$), resulting in a decrease of y_i despite the possible perfect performance (that is, $T_{ij} = M_{ij}$ implying $\Theta_{ij} = 1$) of player i.[14]

Take the tournament (Θ, M) with $M_{ij} = 0$ for players $i, j \in N$. Assume now that $i, k \in N$ are identical players in the tournament (Θ, M), that is, the ith and kth row of Θ and M is equal, respectively, therefore $y(\Theta, M)_i = y(\Theta, M)_k$ from HTV. Let (Θ', M') be a tournament identical to (Θ, M), except for some matches between i and j, that is, $M'_{ij} > 0$. If $\Theta'_{ij} < y(\Theta, M)_i - y(\Theta, M)_j$, which is inevitable if $y(\Theta, M)_i - y(\Theta, M)_j > 1$, then $y(\Theta', M')_i$ will be smaller than $y(\Theta', M')_k$, despite the possible perfect

[14]If Θ'_{ij} coincides with its 'forecast' $y(\Theta, M)_i - y(\Theta, M)_j$ from the tournament (Θ, M) identical to (Θ', M'), but without a comparison between i and j ($\Theta_{ij} = M_{ij} = 0$, providing that the graph of (Θ, M) remains connected), then its optimal value remains unchanged.

performance of i against j. In this case it seems to be unfair that $\phi(\Theta', M')$ ranks k strictly above i, despite the same performance against the common opponents and the perfect result against the others. Obviously, there is no chance for a result better than $\Theta'_{ij} = 1$ if j and k play some matches. It creates an incentive for k to 'impede' its comparison with j. This violation of the property SCM is highlighted in a numerical example.

Example 4.2. *SCM is not satisfied by recursive Buchholz:*[15]

$$\Theta = \begin{pmatrix} 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0.8 & 0 \\ -0.8 & -0.8 & 0 & 0.8 \\ -0.2 & 0 & -0.2 & 0 \end{pmatrix}, \Theta' = \begin{pmatrix} 0 & 0 & 0.8 & 1 \\ 0 & 0 & 0.8 & 0 \\ -0.8 & -0.8 & 0 & 0.8 \\ -1 & 0 & -0.2 & 0 \end{pmatrix};$$

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, M' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix};$$

$$\Lambda = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 5/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{pmatrix} \text{ and } R = \begin{pmatrix} 0.8 \\ 0.8 \\ -0.8 \end{pmatrix}, R' = \begin{pmatrix} 1.8 \\ 0.8 \\ -0.8 \end{pmatrix}.$$

The rating vectors are[16]

$$\phi(\Theta, M) = \begin{pmatrix} 0.6 \\ 0.6 \\ -0.2 \\ -1 \end{pmatrix} \text{ and } \phi(\Theta', M') = \begin{pmatrix} 0.4 \\ 0.6 \\ -0.2 \\ -0.8 \end{pmatrix}.$$

In this case player 1 plays an extra match against player 4, and despite the perfect victory, she loses relative to player 2. However, if player 2 should also play against player 4, she cannot perform better than player 1. Hence it seems to be unfair to rank player 2 above player 1. The reason is that the 'predicted' result of the match between player 1 and 4 is $y(\Theta, M)_1 - y(\Theta, M)_4 = 1.8 > \Theta'_{14} = \max \Theta_{ij} = 1$.

For an exact definition of SCM, see [14]. In the pairwise comparison approach, this problem could theoretically be eliminated if values in Θ are taken from the entire number line, but it is not certainly possible ex ante as these should be defined before the calculation. Note that SCM differs from all the former properties in a basic feature that it permits the modification of both the results matrix Θ and the matches matrix M, not only one of them, like Θ for PRB, BPI or $NNRB$ and M for HOM. Finally, it is necessary to emphasize that SCM is not a relevant property if all players have played the same number of matches, when the results of two players i and j could be compared if and only if they have played against the same opponents, and HTV ensures that if player i has performed better than player j, then recursive Buchholz will reflect it.

5 Recursive Buchholz as an alternative chess ranking

5.1 Chess team tournament rankings

Chess tournaments are often organized in the Swiss-system. It goes for a predetermined number of rounds, and in each round two players compete head-to-head. All players participate in the entire tournament, none are eliminated. The system is used when there are too many players to play a round-robin tournament, consequently, there are pairs of players without a match between them. However, it is more efficient than a knock-out tournament as more matches can be played at the same time due to the low infrastructural requirements.

The two main issues are how to pair the players and how to rank the participants based on their respective results. The pairing method is not discussed in this paper. The principle of a Swiss-system

[15] It is an example from [14].

[16] $y(\Theta, M)$ can be computed by the axiom CMP, and it provides $y(\Theta', M')_2 - y(\Theta', M)_3 = 0.8$, too.

tournament is that a player should play against players with a similar performance. Naturally, in the early rounds this is not possible, the participants are paired according to some exogenous pattern or randomly. For subsequent rounds, the participants are ranked based on their scores and pairs are assigned to be in the same (or close) score group. One rule is that two players cannot play more than once; the others are different adjustments according to the type of sport. However, in chess team championships usually there is no need for them [2, 1]. The pairing procedure is commonly accepted, nevertheless, some proposals have been recommended to improve them by stable matchings [17].

In chess, there are both individual and team tournaments. From an analytical point of view, the latter seems to be preferable, since in individual championships colour allocation has a prominent role; it is not neutral whether the given result is achieved by white or black.[17] In team tournaments a match is played on an even number of boards, and colours are allocated equally. It does not mean that distortions from these are fully eliminated,[18] but it can be more plausible that it does not count.

The next difficulty is caused by different schedules: the opponents of a given competitor strongly influence its performance. It is not a problem in round-robin tournaments, for which [19] has given a full characterization of the score method if the binary beating relation is complete and asymmetric, that is, there are no ties and the intensities of wins are ignored. Among his three axioms, anonymity (ANO) could not be criticized and recursive Buchholz trivially satisfies it, too. The two others, notably positive responsiveness to the beating relation (PRB) and independence of irrelevant matches (IIM) are much more complicated. Recursive Buchholz satisfies PRB if there exists no bridge player: in a 'perfect' Swiss-system tournament, when all teams have played the same, but not too few, number of matches, (intuitively) it has very low probability that bridge players can be found since the whole concept of the pairing algorithm is against it.[19] Nevertheless, in practice byes,[20] late arrivals and unfinished games are present, which complicate the issue. The third property (IIM) is not attractive for tournaments where players have different schedules. Since score consistency (SCC) implies that for ideal round-robin tournaments recursive Buchholz corresponds with the score method, it can be regarded as a modification of the latter for incomplete tournaments. The main drawback of recursive Buchholz, notably that it violates self-consistent monotonicity (SCM), is not a problem if all teams have played the same number of matches, as it has been revealed in Section 4.[21]

Taking into account the aspects above, recursive Buchholz seems to be an acceptable ranking method for Swiss-system chess team tournaments. One of its most attractive feature is simple computation by solving a system of linear equations. The uniqueness of the rating vector requires the connectedness of graph G representing the matches of the championship. Due to the properties of the pairing algorithm, it is achieved relatively fast: for example, in the 18th European Chess Team Championship open section, taken place in Greece 2011 with 38 teams, this was ensured after the third round. It is one of the best possible result since a graph with 38 nodes should have at least 37 edges to be connected, and two rounds include no more than 38 matches.

To apply recursive Buchholz as a ranking method, the results of the matches should be incorporated into the matrix Θ. For this purpose, it is necessary to draw the main features of the official rankings and their connections to recursive Buchholz. A match is played on $2k$ boards, the winner of a game on one board gets 1 (board) point, the loser 0 point, and the draw yields 0.5 for both teams, thus $2k$ board points are allocated between them. The winner team, which achieves more (at least $k+0.5$) board points scores 1 match point, the loser 0, while a draw results in 0.5 match points for each of them.

It is easy to see that match points will result in a lot of ties: for instance, in the 18th European Chess Team Championship, the 38 teams have played 9 rounds with a maximum of 18 match points. To solve this problem, in Swiss-system chess tournaments some tie-break systems are used to rank teams who have the same total number of match points after the last round. It is necessary when prizes are indivisible, such as an official 'champion', or qualification for another tournament. The tie-breaking is a lexicographical order, where the first aspect is the number of scores (match points) and the others are

[17]There are individual chess championships when matches are played in both directions, but they are round-robin tournaments, for which the score method provides an obvious ranking.

[18]For example, it is possible that a team has only one excellent player, when it is a relevant issue. Despite this, according to our knowledge, no official chess ranking method takes colours into account, at most the number of matches with black.

[19]Some simulations are worth to run for different number of participants and rounds to get exact data about it.

[20]A bye is a team, which did not play in a given round due to the odd number of contestants.

[21]However, it does not mean that SCM is not an issue in 'perfect' Swiss-system tournaments. It is true *ex post*, after it has been registered that all teams have played the same number of matches. But *ex ante* it can cause problems by creating incentives not to play a match. We thank Julio González-Díaz for this comment.

various numerical indices based on the results of games have already played.

Definition 5.1. *The basic tie-breaking methods used in a given sequence, until the tie is broken [2, 1] (B_{ij} signs the number of board points team i has scored against team j):*[22]

- *Match points: the sum of match points scored*
  $TB1_i = \sum_{j \in N} |j : B_{ij} \geq k + 0.5| + 0.5 |j : B_{ij} = k|$;
- *Board points: the sum of board points scored*
  $TB2_i = \sum_{j \in N} B_{ij}$;
- *Buchholz points:*[23] *the sum of match points of each opponents*
  $TB3_i = \sum_{j \in N} \chi (B_{ij} + B_{ji} = 2k) \, TB1_j$;
- *Sonneborn-Berger points:*[24] *the sum of match points of each opponent, multiplied by the number of board points achieved against this opponent*
  $TB4_i = \sum_{j \in N} \chi (B_{ij} + B_{ji} = 2k) \, TB1_j B_{ij}$;

where $|S|$ means the number of elements of set S, and χ is the indicator function: $\chi (B_{ij} + B_{ji} = 2k) = 1$ if and only if teams i and j were paired in the tournament and $\chi (B_{ij} + B_{ji} = 2k) = 0$ when there was no match between i and j.

5.2 A variant of recursive Buchholz specified for chess team tournaments

The usual methods for ranking in Swiss-system chess team tournaments are often debated and give strange results in some cases [15]. An example suggests that the official lexicographical orders are not able to account for different schedules properly [9]. The usage of tournament matrices require the transformation of match results into a numerical value Θ_{ij} (and consequently, $\Theta_{ji} = -\Theta_{ij}$), which means some degree of freedom, the method is not exactly defined. In this setting the comparison of two teams can take $4k + 1$ possible values, board points can vary from 0 to $2k$ by 0.5. Only monotonic transformations are considered (winning more board points against the opponent cannot imply a lower value). We propose equal weights for all matches played, since it coincides with the concept of official rankings and there is no cause (at least ex ante) for differentiate between them.[25] The main idea behind recursive Buchholz is that the team that had a harder schedule to achieve the same number of match points should be ranked higher.

Definition 5.2. *A variant of the* recursive Buchholz method specified for a chess team tournaments *is based on a tournament (Θ, M), where:*

1. *The results matrix Θ is filled on the basis of match results;*
2. *Diagonal elements of Θ are zeros;*
3. *Draws (matches with k board points for each team) are represented in Θ by zeros ;*
4. *Other match results are represented in in Θ by a non-decreasing function with respect to the number of winner's board points: $0 < v(k + 0.5) \leq v(k + 1) \cdots \leq v(k + h) \leq \cdots \leq v(2k)$;*
5. *Θ is skew-symmetric: $0 > v(k - 0.5) = -v(k + 0.5) \geq v(k - 1) = -v(k + 1) \geq \cdots \geq v(0) = -v(2k)$;*

[22] The definition of Buchholz and Sonneborn-Berger points for team tournaments is not obvious, they can based on match points [1], or board points as well [2].

[23] Buchholz system was developed by Bruno Buchholz in 1932, but unfortunately we do not know about the original paper. In practice, usually some modifications are applied to avoid the distortions arising from the random pairings in early rounds. For instance, Median Buchholz discards the best and the worst opponents, whereas the FIDE's variant ignores the opponent with the lowest score [1].

[24] We use the definition of [1].

[25] Nevertheless, it seems to be an interesting trial how the weights can be modified after the calculation; for instance, by overrepresenting matches between teams with close ratings, or lowering them in the early rounds. Maybe some iterative procedures can be defined.

6. *Unplayed matches are represented by zeros in* Θ;[26]

7. *The matches matrix* M *is the adjacency matrix of graph* G *corresponding to the tournament:* $M_{ij} = 1$ *if teams* i *and* j *have played against each other, otherwise* 0.

The ranking is determined by sorting the recursive Buchholz rating vector $y_{-1} = \Lambda R_{-1}, y_n = 0$.

Remark 3. *The comparability of teams in* N *ensures the existence and uniqueness of* y.

The fact that the recursive Buchholz method satisfies homogeneous treatment of victories (HTV) means it is neutral for two teams with a common schedule whether they have defeated the stronger or the weaker opponents. At first sight, it is a mistake as it seems to be justifiable to award the beating of strong competitors. However, if two teams scored the same aggregated points, that is, $R_i = R_j$,[27] then a win against a better team is compensated by a loss against a weaker opponent. It is not clear, who should be preferred: a team with a balanced performance (reliable win against underdogs and probable loss against the top) or a team which can cause surprises (sometimes win and sometimes loss against the chances). Similarly, acyclicity (AC) implies the result does not depend on the performance of the defeated opponent, a team could not gain if it win against better teams, but lose against weaker, and vice versa. Tie-breaking rules $TB1$, $TB2$ and $TB3$ satisfy HTV and AC (since they use sums), while $TB4$ is not independent of this characteristic of the results, because the change of the product $TB1_j T_{ij}$ is ambiguous.

Proposition 5.1. *The recursive Buchholz method* generalizes $TB1$ *if all wins are transformed to the same value, that is,* $v(k+0.5) = v(k+1) = \dots = v(k+h) = \kappa = \dots = v(2k) = \kappa$, *where* κ *could be chosen arbitrarily due to Scale-invariance (SI). The original ranking based on match points* $TB1$ *is adjusted through the Laplacian matrix* $L(G)$.

Proof.

$$R_i = \sum_{j=1}^{n} M_{ij} B_{ij} = \kappa (TB1_i - d_i),$$

since wins are represented by κ, draws by 0 and losses by $-\kappa$. Therefore R_i could be calculated as a linear function of $TB1_i$ with a positive slope. □

Remark 4. *The above variant has no consistent*[28] *triads; if team* i *scores* $k+0.5$ *board points against* j, *and* j *scores* $k+0.5$ *against* ℓ, *then* i *is not able to beat* ℓ *with a proper margin. In the calculation form of [14], the relation with Buchholz points (*$TB3$*) is clear: the scores vector* s *give a ranking coinciding with* R *and* $TB1$ *(the division by the total number of matches is irrelevant since all teams have played the same number of matches).*

Proposition 5.2. *The recursive Buchholz method* generalizes $TB2$ *if wins are transformed as follows:* $v(k+0.5) = \kappa < v(k+1) = 2\kappa < \dots < v(k+h) = 2h\kappa < \dots < v(2k) = 2k\kappa$, *where* κ *could be chosen arbitrarily due to the property SI. The original ranking based on board points* $TB2$ *is adjusted through the Laplacian matrix* $L(G)$.

Proof.

$$R_i = \sum_{j=1}^{n} M_{ij} B_{ij} = \kappa (TB2_i - 2k d_i),$$

therefore R_i could be calculated as a linear function of $TB2_i$ with a positive slope. □

Remark 5. *This variant has a consistent triad if team* i *scores* $k+1$ *board points against* j *and* j *scores* $k+0.5$ *against* ℓ *means that* i *scores* $k+1.5$ *against* ℓ.

[26] However, there is a difference between draws and unplayed matches in matrix M, therefore in $L(G)$.

[27] It will be showed later that it can correspond both to $TB1_i = TB1_j$ or to $TB2_i = TB2_j$.

[28] In the tournament approach, consistency could be defined as for the pairwise comparison matrix setting: a tournament matrix is consistent if and only if the optimal value of the weighted least squares objective function is its minimum of 0. It is the case in Example 4.2 and from CMP it is obvious that all tournaments with only critical matches are consistent.

Propositions 5.1 and 5.2 an idea how to transform the match results into a tournament matrix θ. The generalized $TB1$ and $TB2$ could be considered as two extreme cases, it is not worth to regard a more 'tilted' v than for $TB2$ as it still significantly awards the winner, while generalized $TB1$ applies a flat function v. Moreover, it is practical to prefer the latter because the other variant favours teams which play weakly in the early rounds and then meet underdogs, who may be beaten by a higher margin.[29] Nevertheless, the official lexicographical orders also use $TB1$ as the main aspect, and the number of match points forms the foundation of the pairing algorithm. The transition between the two extremes can be regulated by a parameter $1 \leq \gamma \in \mathbb{R}$, inversely representing the importance of board points:

$$v(k+0.5) = \kappa < v(k+1) = \left(1 + \frac{1}{\gamma}\right)\kappa < \dots < v(k+h) = h\left(1 + \frac{1}{\gamma}\right)\kappa < \dots < v(2k) = k\left(1 + \frac{1}{\gamma}\right)\kappa,$$

where $\gamma = 1$ is for generalized $TB2$ and $\gamma \to \infty$ is for generalized $TB1$. Regarding the two other tie-breaking rules, $TB3$ is implicitly contained by generalized $TB1$ (it is a new argument for choosing a large γ) and $TB4$ is worth to ignore, since it does not satisfy homogeneous treatment of victories (HTV) and acyclicity (AC) by overrepresenting wins against strong opponents. γ should be determined with respect to these arguments.

6 Conclusions

This paper examines an alternative method to determine a ranking for tournaments. It was defined by [14] as recursive Buchholz, but it is not an entirely new concept, since it has strong links to the incomplete weighted version of $LLSM$ method based on incomplete $LLSM$ by [4].[30] We have presented a common framework to unify these approaches and have proven four new properties of the method, namely, SI, AC, CMP and $WPRB$. The first two is the consequence of our setting (Θ, M) for a tournament, whereas CMP is a bit outlier since it refers to the rating vector, not only to the ranking derived from it. $WPRB$ reveals the coherence of BPI and $NNRB$, for which an alternative proof is given. Finally, a variant of the method have been defined for chess team tournaments, suggesting it as a way to improve the currently used lexicographical orders and preserve simple calculation as a solution of a system of linear equations. This approach can be extended to other sport championships.

The major weakness of recursive Buchholz is that it violates SCM, therefore further research is needed in order to modify it for cases when the players have not played the same number of matches (the diagonal elements of the Laplacian matrix corresponding to the weighted undirected graph G are different). Since the rating vector is based on a Laplacian matrix, some synergies with graph theory are possible.

Acknowledgements

The author is thankful to Sándor Bozóki and Julio González-Díaz for helpful discussion. This work was supported by OTKA grant K-77420.

References

[1] *Handbook of FIDE. D.II.02. Regulations for specific competitions, Chess olympiad, Olympiad pairing rules. Approved by the 1994 and 1998 Executive Councils, amended by the 2006 Presidential Board and the 2007 and 2009 Executive Council.* `http://www.fide.com/component/handbook/?id=95&view=article`, 2009.

[29] To highlight this issue, take two teams i and j with the similar number of match points, which was achieved by performing stronger in the early rounds (on the 'inner circle', for i) and playing better in the subsequent rounds (on the 'outer circle', for j), respectively. Then, j had probably weaker (notably, teams with lower average match points) opponents due to the pairing method. In this case, $TB1$ and especially $TB2$ awards team j, while team i benefits from $TB3$ and $TB4$ is ambiguous. Intuitively, team i should be preferred because of stronger opponents. The example of [9] really shows that a version of recursive Buchholz is in fact closer to a ranking based on $TB3$ than to $TB1$.

[30] However, we do not think it is worth to debate about the origin of the method.

[2] *European Chess Union. Tournament Rules 2012.* `http://europechess.net/index.php?option=com_content&view=article&id=9&Itemid=15`, 2012.

[3] S. Bozóki, L. Csató, L. Rónyai, and J. Tapolcai. Robust peer review decision process. Manuscript, 2012.

[4] S. Bozóki, J. Fülöp, and L. Rónyai. On optimal completion of incomplete pairwise comparison matrices. *Mathematical and Computer Modelling*, 52(1-2):318–333, 2010.

[5] M. Brozos-Vázquez, M. A. Campo-Cabana, J. C. Díaz-Ramos, and J. González-Díaz. Ranking participants in tournaments by means of rating functions. *Journal of Mathematical Economics*, 44(11):1246–1256, 2008.

[6] A.T.W. Chu, R.E. Kalaba, and K. Spingarn. A comparison of two methods for determining the weight belonging to fuzzy sets. *Journal of Optimization Theory and Applications*, 27(4):531–538, 1979.

[7] G. Crawford and C. Williams. Analysis of subjective judgment matrices. *The Rand Corporation, Office of the Secretary of Defense USA, R-2572-AF*, 1980.

[8] G. Crawford and C. Williams. A note on the analysis of subjective judgment matrices. *Journal of Mathematical Psychology*, 29(4):387–405, 1985.

[9] L. Csató. Ranking by pairwise comparisons for Swiss-system tournaments. *Central European Journal of Operations Research*, DOI: 10.1007/s10100-012-0261-8, First published online: 27 July, 2012.

[10] J.G. De Graan. Extensions of the multiple criteria analysis method of T.L. Saaty. Presented at EURO IV Conference, Cambridge, UK, July 22-25, 1980.

[11] J. Fülöp. A method for approximating pairwise comparison matrices by consistent matrices. *Journal of Global Optimization*, 42(3):423–442, 2008.

[12] S. I. Gass. Tournaments, transitivity and pairwise comparison matrices. *The Journal of the Operational Research Society*, 49(6):616–624, 1998.

[13] P.T. Harker. Incomplete pairwise comparisons in the analytic hierarchy process. *Mathematical Modelling*, 9(11):837–848, 1987.

[14] R. Hendrickx, E. Lohmann, and J. González-Diaz. Paired comparisons analysis: an axiomatic approach to rankings in tournaments. CentER Working Paper Series 2011-116, Tilburg University, 2011.

[15] V. M. Jeremic and Z. Radojicic. A new approach in the evaluation of team chess championships rankings. *Journal of Quantitative Analysis in Sports*, 6(3):Article 7, 2010.

[16] G. Kéri. On qualitatively consistent, transitive and contradictory judgment matrices emerging from multiattribute decision procedures. *Central European Journal of Operations Research*, 19(2):215–224, 2011.

[17] E. Kujansuu, T. Lindberg, and E. Mäkinen. The stable roommates problem and chess tournament pairings. *Divulgaciones Matemáticas*, 7(1):19–28, 1999.

[18] O. Perron. Zur Theorie der Matrices. *Mathematische Annalen*, 64(2):248–263, 1907.

[19] A. Rubinstein. Ranking the participants in a tournament. *SIAM Journal on Applied Mathematics*, 38(1):108–111, 1980.

[20] T.L. Saaty. *The analytic hierarchy process: planning, priority setting, resource allocation.* McGraw-Hill International Book Co., 1980.