Properties of risk capital allocation methods

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Abstract

In finance risk capital allocation is an important question from both theoretical and practical point of view. How to share risk of a portfolio among its subportfolios? How to reserve capital in order to hedge existing risk and how to assign this to different business units?

We use an axiomatic approach to examine risk capital allocation, i.e. we call for fundamental properties of the methods. The starting point of our article is the theorem of Csóka and Pintér (2010) who showed that the requirements of Core Compatibility, Equal Treatment Property and Strong Monotonicity are irreconcilable given that risk is quantified by a coherent measure of risk.

In this article we examine these requirements using analytic and simulation tools. We examine allocation methods used in practice and also ones which are theoretically interesting. The main result of the article is that the problem raised by Csóka and Pintér (2010) is also relevant in practical applications, so it is not only a theoretical problem. We also believe that through the characterization of the examined methods this article can be a useful guide for practitioners.

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1 Introduction

Proper measurement of risk is essential for banks, insurance companies, portfolio managers and other units exposed to financial risk. In this paper we consider the class of coherent measures of risk\(^1\) (Artzner et al., 1999), which are defined by four natural axioms: Monotonicity, Translation Invariance, Positive Homogeneity and Subadditivity. Coherent measures of risk are broadly accepted in the literature, for a further theoretical foundation of the notion see Csóka et al. (2007) and Acerbi and Scandolo (2008).

Subadditivity captures the notion of diversification: if a financial unit (portfolio) consists of more subunits (subportfolios) the sum of the risks of the subunits is at least as much as the risk of the whole unit. The saving of risk should be assigned to the subunits. This process is called risk capital allocation and has the following applications (see for instance Denault (2001); Kalkbrenner (2005); Valdez and Chernih (2003); Buch and Dorfleitner (2008); Homburg and Scherpereel (2008); Kim and Hardy (2009); Csóka et al. (2009)):

1. financial institutions divide their capital reserves between business units,
2. strategic decision making regarding new business lines,
3. product pricing,
4. (individual) performance measurement,
5. forming risk limits,
6. risk sharing among insurance companies.

Using methods of cooperative game theory Csóka and Pintér (2010) showed that considering a general (not specific, only defined by the axioms) coherent measure of risk there is no risk capital allocation method satisfying three fairness requirements at the same time: Equal Treatment Property, Core Compatibility and Strong Monotonicity. The description of the properties is as follows.

\(^1\)As a further reading on risk measures we suggest Krokhmala et al. (2011).
Equal Treatment Property requires that subunits having the very same risk characteristics (alone and also when joining any subset other subunits) are treated the same way. It is even possible that using a method that does not satisfy Equal Treatment Property might have legal consequences.

Strong Monotonicity requires that if the risk characteristics of a subunit is not decreasing, then its allocated risk capital should not decrease either. Hence - as an incentive compatibility notion - the subunits are not motivated to increase their risk characteristics.

Core Compatibility ensures that the risk of the main unit is allocated in such a way that no subset (coalition) of subunits is allocated more risk capital than it would be implied by its own risk. Otherwise the given subunit group would strongly argue against the allocation and block it, or at least feel that it is not fair.

As there is no risk capital allocation method that satisfies all the above given properties at the same time when risk is calculated by a general coherent measure of risk, we analyze seven allocation methods to find out how are they related to the three fairness requirements. The seven methods are: Activity based method (Hamlen et al. (1977)), Beta method (see for instance Homburg and Scherpereel (2008)), Incremental method (see for instance Jorion (2007)), Cost gap method (Driessen and Tijs (1986)), Euler method (see for instance Buch and Dorflheitner (2008)), Shapley method (Shapley, 1953) and the Nucleolus method (Schmeidler, 1969).

We use two different approaches. First, we discuss analytically which properties are fulfilled by the different methods. Our results are as follows: from the seven methods, two methods well-known from game theory, the Shapley and the Nucleolus method showed the best performance, as both of them satisfy two out of the three properties. In case of the Shapley method we have to give up Core Compatibility, while Strong Monotonicity is not fulfilled by the Nucleolus method.

Then, we analyze Core Compatibility for all the methods using simulation. We examine how often is it fulfilled for normal and also for fat tailed return distributions. Our results are presented in the tables of Section 4. Since Csóka and Pintér (2010) shows that for risk capital allocation situations the only allocation method satisfying Equal Treatment Property, Strong Monotonicity and Pareto Optimality is the Shapley method, violation of Core Compatibility in case of the Shapley method needs a special attention. Our simulation results show that Core Compatibility is violated by the Shapley method in a significant number of the cases in every examined simulation setting. Hence we conclude that the theoretical problem raised by Csóka and Pintér (2010), the impossibility of the existence of an ideal risk capital allocation method is also a relevant practical problem. Thus
all of the examined methods have their own advantages and disadvantages. Practitioners have to decide on the applied method in every single situation, considering which method fits best the actual setting. Our analytical and simulation results are intended to help this decision.

The setup of our paper is as follows: in the next section we define the notions used in our analysis. In Section 3 we introduce the seven examined methods, while we present our results in the last section.

2 Preliminaries

In this paper, for the sake of simplicity, we will assume that the subunits can be described by random variables defined on a finite probability field \((\Omega, \mathcal{M}, P)\). We will denote random variables as \(X : \Omega \to \mathbb{R}\). As we have a finite number of states, we can think of the random variables as vectors (their components are the elements of \(\Omega\)). The set of the random variables defined on the fixed finite probability field \((\Omega, \mathcal{M}, P)\) is denoted by \(\mathcal{X}\).

We analyze the following financial situation. A financial unit (for instance a bank, a portfolio, etc.) consists of a finite number of subunits. The set of the subunits are denoted by \(N = \{1, 2, \ldots, n\}\). Every financial unit can be described using a random variable, for subunit \(i\) it is random variable \(X_i\). In other words, the vector \(X_i\) shows the possible profits of subunit \(i\) for a given time period (negative values mean losses). We are also interested in subsets of subunits, a coalition \(S \subseteq N\) is described by \(X_S = \sum_{i \in S} X_i\), where the value of the empty sum is the 0 function (vector).

The function \(\rho : \mathcal{X} \to \mathbb{R}\) is called a measure of risk. A measure of risk assigns a number to every random variable, it defines the risk of every coalition. This notion of risk can also be seen as the minimum level of capital (cash) to be added to the portfolio. In this paper we restrict ourselves to coherent measures of risk (Artzner et al., 1999).

In our simulations we use the \(k\)-Expected Shortfall measure of risk (Acerbi and Tasche, 2002), which is the average of the worst \(k\) percentage of the losses. At last, the notion risk capital allocation situation, RCAS will be used for the vectors describing the financial subunits and the measure of risk together, so \(X^\rho_N = \{N, \{X_i\}_{i \in N}, \rho\}\) is a risk capital allocation situation. The class of risk capital allocation situations with the set of subunits \(N\) will be denoted by \(\text{RCAS}_N\).

Example 2.1. Consider the \(X^\rho_N\) risk capital allocation situation where on the probability field \((\Omega, \mathcal{M}, P)\) the set \(\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}\), \(\mathcal{M}\) denotes all subsets of \(\Omega\) and \(P\) is such that \(P(\{\omega_1\}) = P(\{\omega_2\}) = P(\{\omega_3\}) = P(\{\omega_4\}) = \frac{1}{4}\). We have three subunits, \(N = \{1, 2, 3\}\), and the applied measure of risk \(\rho\) is the
25%—Expected Shortfall, which in this case (and for all $k \in [0, 25]$) equals the maximum loss. The vectors $X_i$ are shown in the following table.

<table>
<thead>
<tr>
<th>$\Omega/X_S$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_{{1,2}}$</th>
<th>$X_{{1,3}}$</th>
<th>$X_{{2,3}}$</th>
<th>$X_{{1,2,3}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>-10</td>
<td>-10</td>
<td>0</td>
<td>-20</td>
<td>-10</td>
<td>-10</td>
<td>-20</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>-3</td>
<td>-4</td>
<td>-100</td>
<td>-7</td>
<td>-104</td>
<td>-107</td>
<td></td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>-6</td>
<td>0</td>
<td>-99</td>
<td>-6</td>
<td>-105</td>
<td>-99</td>
<td>-105</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>0</td>
<td>-6</td>
<td>-99</td>
<td>-6</td>
<td>-99</td>
<td>-105</td>
<td>-105</td>
</tr>
<tr>
<td>$\rho(X_S)$</td>
<td>10</td>
<td>10</td>
<td>100</td>
<td>20</td>
<td>105</td>
<td>105</td>
<td>107</td>
</tr>
</tbody>
</table>

Table 1: The risk capital allocation situation $X^\rho_N$.

The lowermost row contains the risk of each coalition. Note that a significant diversification effect arises when the three subunit groups form a coalition:

$$\rho(X_N) = 107 < 120 = \sum_{i \in N} \rho(X_i).$$

As in the above example, two questions arise in most risk capital allocation situations. How to allocate the diversification benefits among the subunits? What properties should the applied allocation method have?

The function $\varphi : \text{RCAS}_N \rightarrow \mathbb{R}^N$ will be called risk capital allocation method. A risk capital allocation method determines the necessary capital reserve for each subunit in the particular risk capital allocation situation. We define three properties of an allocation method.

**Definition 2.2.** The risk capital allocation method $\varphi$ satisfies

1. **Equal Treatment Property (ETP),** if for all risk capital allocation situation $X^\rho_N \in \text{RCAS}_N$ and arbitrary subunits $i, j \in N$, and in case of any subunit coalition $S \subseteq N \setminus \{i, j\}$ the equations $\rho(X_{S\cup\{i\}}) - \rho(X_S) = \rho(X_{S\cup\{j\}}) - \rho(X_S)$ imply that $\varphi_i(X^\rho_N) = \varphi_j(X^\rho_N),$

2. **Strong Monotonicity (SM),** if for all risk capital allocation situation $X^\rho_N, Y^\rho_N \in \text{RCAS}_N$ and arbitrary subunit $i \in N$ for all $S \subseteq N$ the inequalities $\rho(X_{S\cup\{i\}}) - \rho(X_S) \leq \varrho(Y_{S\cup\{i\}}) - \varrho(Y_S)$ imply that $\varphi_i(X^\rho_N) \leq \varphi_i(Y^\rho_N),$

3. **Core Compatibility (CC),** if for all risk capital allocation situation $X^\rho_N \in \text{RCAS}_N$ the equality $\rho(X_N) = \sum_{i \in N} \varphi_i(X^\rho_N)$ holds, and for all coalition $S \subseteq N$ we have that $\rho(X_S) \geq \sum_{i \in S} \varphi_i(X^\rho_N).$
The properties can be explained as follows.

Equal Treatment Property makes sure that similar subunits are treated equally, that is if two subunits make the same risk contribution to all coalitions not containing them (also to the empty set, meaning that their stand-alone risk is the same), then the rule should allocate the very same risk capital to the two subunits.

Strong Monotonicity requires that if the contribution of risk of subunit to all the coalitions of subunits is not decreasing, then its allocated risk capital should not decrease either. Hence, as an incentive compatibility notion, the subunits are not motivated to increase their contribution of risk.

Core Compatibility is satisfied if a risk allocation is stable: the risk of the main unit is fully allocated and no coalitions of the subunits would like to object their allocated risk capital by saying that their risk is less than the sum of the risks allocated to the members of the particular coalition.

The following theorem holds.

Theorem 2.3 (Csóka and Pintér (2010)). Considering a general coherent measure of risk there is no risk capital allocation method meeting the conditions of Core Compatibility, Strong Monotonicity and Equal Treatment Property at the same time.

In other words, if an arbitrary coherent measure of risk is applied, then in general Core Compatibility, Strong Monotonicity and Equal Treatment Property can not be fulfilled at the same time. Our aim was to find out how relevant this problem is in practice considering return distributions having empirically observable characteristics. The question is answered in Section 4.

3 Risk capital allocation methods

In this section we will present seven risk capital allocation methods that we examine later in this paper. During the introduction of the first four methods we will follow Homburg and Scherpereel (2008). The fifth method, the Euler method has three different versions. The last two methods, the Shapley (Shapley, 1953) and the Nucleolus method (Schmeidler, 1969) are well-known in game theory.

3.1 Activity based method

The method was first introduced by Hamlen et al. (1977). It allocates the joint risk to the subunits in proportion to their own risk. For an arbitrary risk
capital allocation situation \( X_N^\rho \in \text{RCAS}_N \) and subunit \( i \in N \) the Activity based method allocates

\[
\varphi_{i}^{AB}(X_N^\rho) = \frac{\rho(X_i)}{\sum_{j \in N} \rho(X_j)} \rho(X_N).
\]

A really serious drawback of the method is that it does not consider the dependence structure of the subunits, that is it does not allocate smaller risk capital to those subunits that are hedging the others.

### 3.2 Beta method

Let \( X_N^\rho \in \text{RCAS}_N \) be a risk capital allocation situation and let \( \text{Cov}(X_i, X_N) \) denote the covariance between the random variables describing the financial position of \( i \in N \) subunit and the whole financial unit. It is well known that the beta of unit \( i \) is calculated as

\[
\beta_i = \frac{\text{Cov}(X_i, X_N)}{\text{Var}(X_N)}.
\]

For an arbitrary risk capital allocation situation \( X_N^\rho \in \text{RCAS}_N \) and subunit \( i \in N \) the Beta method allocates

\[
\varphi_i^B(X_N^\rho) = \frac{\beta_i}{\sum_{j \in N} \beta_j} \rho(X_N).
\]

We have to mention that the Beta method is not always defined. If for instance there is no aggregate risk, then all of the betas are zeros, so the above given formula can not be applied.

### 3.3 Incremental method

The Incremental method (see for instance Jorion (2007)) defines the risk capital allocated to a subunit in proportion to its individual “risk increment”, which is its contribution of risk to the main unit. For any risk capital allocation situation \( X_N^\rho \in \text{RCAS}_N \) and subunit \( i \in N \) the Incremental method allocates

\[
\varphi_i^N(X_N^\rho) = \frac{\rho(X_N) - \rho(X_{N\setminus\{i\}})}{\sum_{j \in N} \rho(X_N) - \rho(X_{N\setminus\{j\}})} \rho(X_N).
\]

We also have to mention that this method can not always be calculated, for example when we consider the risk capital allocation situation generated by the identity matrix multiplied by minus one for all subunit \( i \) we have that \( \rho(X_N) - \rho(X_{N\setminus\{i\}}) = 0 \), so the above given formula can not be applied.
3.4 Cost gap method

After a smaller amendment on the Incremental method we get the Cost gap allocation rule, defined by [Driessen and Tijs (1986)]. First of all, let us define for each subunit \( i \in N \) its smallest cost gap (\( \gamma_i \)) as follows:

\[
\gamma_i = \min_{\emptyset \neq S \subseteq N, i \in S} \left| \rho(X_S) - \sum_{j \in S} (\rho(X_N) - \rho(X_{N \setminus \{j\}})) \right|.
\]

The expression after the minimum is the cost gap of coalition \( S \), showing the difference between the risk of the coalition and the contributions of its members to the grand coalition. It is in fact the non-separable risk (cost) of coalition \( S \). If - as we shall see in the Cost gap method - we modify the Incremental method in such a way that the non-separable risk of the grand coalition is shared in proportional to the cost gap of each player, then every subunit would like to join the coalition in which they have the lowest cost gap. For any risk capital allocation situation \( X^\rho_N \in \text{RCAS}_N \) and subunit \( i \in N \) the Cost gap method is defined formally as follows.

If \( \sum_{j \in N} \gamma_j = 0 \), then

\[
\varphi_i^{CG}(X^\rho_N) = \rho(X_N) - \rho(X_{N \setminus \{i\}}),
\]

otherwise

\[
\varphi_i^{CG}(X^\rho_N) = \rho(X_N) - \rho(X_{N \setminus \{i\}}) + \frac{\gamma_i}{\sum_{j \in N} \gamma_j} \left( \rho(X_N) - \sum_{j \in N} (\rho(X_N) - \rho(X_{N \setminus \{j\}})) \right).
\]

3.5 Euler method (marginal contribution to risk)

The Euler method, also called gradient method or marginal contribution to risk, is a popular method both in the literature and in practice. Despite this popularity the method is not canonically defined, that is, different versions of the Euler method are used in the literature. In this paper we introduce a notion of Euler method which is quite general, and reflects the main intuitions behind the method.

For any risk capital allocation situation \( X^\rho_N \in \text{RCAS}_N \) and subunit \( i \in N \) we define the Euler as follows:

\[
\varphi_i^E(X^\rho_N) = \rho'(X_N, X_i),
\]
where $\rho'(X_N, X_i)$ is the directional derivative of $\rho$ at $X_N$ along $X_i$. In case the directional derivative of $\rho$ at $X_N$ exists along each $X_i, i \in N$, the Euler method is defined.

In some papers the Euler method is used in a different, more restrictive way. The derivatives are not directional derivatives, but partial derivatives. In this case for any risk capital allocation situation $X_N^\rho \in \text{RCAS}_N$ and subunit $i \in N$ the allocated risk by the Euler method is the following:

$$\rho_i'(X_N) = \partial_{\alpha_i} \left( \sum_{i \in N} \alpha_i X_i \right) \big|_{\alpha_i=1},$$

where $\rho_i'(X_N)$ is the partial derivative of the measure of risk $\rho$ at $X_N$ by $\alpha_i$.

Notice, that if for all $i \in N$ the measure of risk $\rho$ is directionally differentiable at $X_N$ along both $X_i$ and $-X_i$, then $\rho_i'(X_N)$ is defined. (Buch and Dorfleitner, 2008) show that in this case the Euler method satisfies Core Compatibility. However, many coherent measures of risk fail to meet this differentiability condition, for instance (as we will show it in Example 4.5) in general the Expected Shortfall does not meet this condition.

Moreover, by the partial derivative definition the Euler method coincides with the Aumann-Shapley value, and it can be extended to cases where it is not defined in a way that it preserves Core Compatibility (Boonen et al., 2010); naturally this extension differs from the one (directional derivative) we use in this paper.

### 3.6 Shapley method

The Shapley method (Shapley, 1953) defines the risk capital allocated to the subunits as follows. For an arbitrary risk capital allocation situation $X_N^\rho \in \text{RCAS}_N$ and subunit $i \in N$

$$\varphi_i^{Sh}(X_N^\rho) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left( \rho(X_{S \cup \{i\}}) - \rho(X_S) \right),$$

where $|S|$ denotes the number of elements in $S$.

### 3.7 Nucleolus method

The Nucleolus method (Schmeidler, 1969) defines the risk capital allocated to the subunits as follows. For an arbitrary risk capital allocation situation $X_N^\rho \in \text{RCAS}_N$ and subunit $i \in N$

$$\varphi_i^{Nc}(X_N^\rho) = \{ x \in I(X_N^\rho) : E(x) \geq_{\text{lex}} E(y) \text{ for all } y \in I(X_N^\rho) \},$$
where \( I(X_N^\rho) = \{ x \in \mathbb{R}^N : \sum_{j \in N} x_j = \rho(X_N) \text{ for all } j \in N : x_j \leq \rho(X_j) \} \), 
\( E(x) = (\cdots \geq \rho(X_S) - \sum_{i \in S} x_i \geq \cdots)_{S \subset N} \), and \( \geq_{\text{lex}} \) denotes the lexicographic ordering.

We conclude the section with the continuation of Example 2.1

Example 3.1 (Example 2.1 continued.). Table 2 shows allocated risk capital using the above given seven methods for the risk capital allocation situation described in Example 2.1

<table>
<thead>
<tr>
<th>Method/Subunit</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Activity based</td>
<td>8,9167</td>
<td>8,9167</td>
<td>89,1667</td>
</tr>
<tr>
<td>Beta</td>
<td>-8,7390</td>
<td>-8,2969</td>
<td>124,0359</td>
</tr>
<tr>
<td>Incremental</td>
<td>2,3516</td>
<td>2,3516</td>
<td>102,2967</td>
</tr>
<tr>
<td>Cost gap</td>
<td>6,4138</td>
<td>6,4138</td>
<td>94,1724</td>
</tr>
<tr>
<td>Euler</td>
<td>3</td>
<td>4</td>
<td>100</td>
</tr>
<tr>
<td>Shapley</td>
<td>6,5</td>
<td>6,5</td>
<td>94</td>
</tr>
<tr>
<td>Nucleolus</td>
<td>6</td>
<td>6</td>
<td>95</td>
</tr>
</tbody>
</table>

Table 2: The allocated risk capital of Example 2.1 using the different methods.

In this example, in case of the Beta method the betas are respectively \(-0.0817, -0.0775\) and 1.1592, while in case of the Cost gap method the gammas equal 8, 8 and 13, respectively.

4 Properties of the different allocation methods

In this section we present our analytical and simulation results. It is important that we use a general coherent measure of risk, that is we test the fulfillment of the properties for any coherent measure of risk.

4.1 Analytic results

Table 3 summarizes our analytical results.

The interpretation of the table is as follows: the \( \checkmark \) sign means that the method in the given row fulfills the property of the given column. Before we detail our results, we note that the result of Csóka and Pintér (2010) (see Theorem 2.3) implies that none of the methods fulfill all of the properties.
<table>
<thead>
<tr>
<th>Method / Axiom</th>
<th>ETP</th>
<th>SM</th>
<th>CC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Activity based</td>
<td>√</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beta</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Incremental</td>
<td>√</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost gap</td>
<td>√</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Euler</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shapley</td>
<td>√</td>
<td>√</td>
<td></td>
</tr>
<tr>
<td>Nucleolus</td>
<td>√</td>
<td></td>
<td>√</td>
</tr>
</tbody>
</table>

Table 3: Properties of the considered methods

Let us move to the detailed results. The definition of the Activity based method implies that it satisfies Equal Treatment Property and as our simulation results will show, it is not Core Compatible. Regarding Strong Monotonicity we provide the following example.

**Example 4.1.** Let us consider risk capital allocation situations $X^\rho_N$ and $Y^\rho_N$, where for both cases in the probability field $(\Omega, \mathcal{M}, P)$ we have that $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{M}$ denotes all subsets of $\Omega$, and $P$ is such that $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$, we have two subunits, $N = \{1, 2\}$, and the applied measure of risk, $\rho$ is the 50%-Expected Shortfall (the maximum loss again). The vectors $X_i$ and $Y_i$ and their sums are shown in Table 4.

<table>
<thead>
<tr>
<th>$\Omega/X_S$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_{(1,2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>0</td>
<td>-11</td>
<td>-11</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>-10</td>
<td>0</td>
<td>-10</td>
</tr>
<tr>
<td>$\rho(X_S)$</td>
<td>10</td>
<td>11</td>
<td>11</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Omega/Y_S$</th>
<th>$Y_1$</th>
<th>$Y_2$</th>
<th>$Y_{(1,2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>0</td>
<td>-20</td>
<td>-20</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>-9</td>
<td>0</td>
<td>-9</td>
</tr>
<tr>
<td>$\rho(Y_S)$</td>
<td>9</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 4: The risk capital allocation situations $X^\rho_N$ and $Y^\rho_N$.

Here $\rho(X_1) > \rho(Y_1)$ and $\rho(X_N) - \rho(X_2) = 0 = \rho(Y_N) - \rho(Y_2)$, but $\varphi^A_B(X^\rho_N) = \frac{10}{21}11 < \frac{20}{21}20 = \varphi^A_B(Y^\rho_N)$, so the Activity based method does not satisfy Strong Monotonicity.

Beta method does not fulfill any of the analyzed axioms. Example 3.1 shows that it does not satisfy Equal Treatment Property, as it allocates different amounts of capital on subunit 1 and 2. The simulation results will also show that it is not Core Compatible, while regarding Strong Monotonicity we provide the following example:

**Example 4.2.** If we consider the situation given in Example 3.1, then $\rho(X_1) > \rho(Y_1)$ and $\rho(X_N) - \rho(X_2) = 0 = \rho(Y_N) - \rho(Y_2)$, but $\varphi^B(X^\rho_N) = -110 < -16.36 = \varphi^B(Y^\rho_N)$, so the Beta method does not fulfill Strong Monotonicity.
In case of the Incremental method it can also be seen from the defining formula that it satisfies Equal Treatment Property and the simulation results will also show that it is not Core Compatible. We examine Strong Monotonicity in the following example:

**Example 4.3.** Let us consider risk capital allocation situations \( X_\rho^N \) and \( Y_\rho^N \), where for both cases in the probability field \((\Omega, M, P)\) we have that \( \Omega = \{\omega_1, \omega_2\} \), \( M \) denotes all subsets of \( \Omega \), and \( P \) is such that \( P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2} \), we have two subunits, \( N = \{1, 2\} \), and the applied measure of risk, \( \rho \) is the 50%-Expected Shortfall (the maximum loss again). The vectors \( X_i \) and \( Y_i \) and their sum are shown in Table 5.

<table>
<thead>
<tr>
<th>( \Omega/X_S )</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_{(1,2)} )</th>
<th>( \Omega/Y_S )</th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
<th>( Y_{(1,2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 )</td>
<td>-2</td>
<td>-9</td>
<td>-11</td>
<td>( \omega_1 )</td>
<td>-2</td>
<td>-7</td>
<td>-9</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>-9</td>
<td>0</td>
<td>-9</td>
<td>( \omega_2 )</td>
<td>-9</td>
<td>0</td>
<td>-9</td>
</tr>
<tr>
<td>( \rho(X_S) )</td>
<td>9</td>
<td>9</td>
<td>11</td>
<td>( \rho(Y_S) )</td>
<td>9</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 5: Risk capital allocation situations \( X_\rho^N \) and \( Y_\rho^N \).

Here \( \rho(X_1) = \rho(Y_1) \) and \( \rho(X_N) - \rho(X_2) > \rho(Y_N) - \rho(Y_2) \), but \( \varphi^N(X^N_N) = \frac{11}{2} < 9 = \varphi^N(Y^N_N) \), so the Incremental method does not satisfy Strong Monotonicity.

The Cost gap method is very similar to the Incremental method and in this case fulfillment of the Equal Treatment Property can also be seen from the defining formula and the simulation results also show that it is not Core Compatible. We examine Strong Monotonicity in the following example:

**Example 4.4.** Let us consider risk capital allocation situations \( X_\rho^N \) and \( Y_\rho^N \), where for both cases in the probability field \((\Omega, M, P)\) we have that \( \Omega = \{\omega_1, \omega_2\} \), \( M \) denotes all subsets of \( \Omega \), and \( P \) is such that \( P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2} \), we have two subunits, \( N = \{1, 2\} \), and the applied measure of risk, \( \rho \) is the 50%-Expected Shortfall (the maximum loss again). The vectors \( X_i \) and \( Y_i \) and their sum are shown in Table 6.

<table>
<thead>
<tr>
<th>( \Omega/X_S )</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_{(1,2)} )</th>
<th>( \Omega/Y_S )</th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
<th>( Y_{(1,2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 )</td>
<td>-2</td>
<td>-9</td>
<td>-11</td>
<td>( 5\omega_1 )</td>
<td>-9</td>
<td>-1</td>
<td>-11</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>-9</td>
<td>0</td>
<td>-9</td>
<td>( \omega_2 )</td>
<td>0</td>
<td>-11</td>
<td>-11</td>
</tr>
<tr>
<td>( \rho(X_S) )</td>
<td>9</td>
<td>9</td>
<td>11</td>
<td>( \rho(Y_S) )</td>
<td>9</td>
<td>11</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 6: Risk capital allocation situations \( X_\rho^N \) and \( Y_\rho^N \).
If we consider the situation given in Example 4.3, then $\rho(X_1) = \rho(Y_1)$ and $\rho(X_N) - \rho(X_2) = 2 = \rho(Y_N) - \rho(Y_2)$, but $\phi^{CG}_1(X_N) = \frac{1}{2} > \frac{2}{3} = \phi^{CG}_1(Y_N)$ so the Cost gap method does not fulfill Strong Monotonicity.

In case of the Euler method it was showed by Buch and Dorfleitner (2008) that for a differentiable coherent measure of risk the Euler method is Core Compatible, but does not always fulfill Equal Treatment Property (see also Example 2.1 for the violation of ETP). However, generally the method does not satisfy any of the axioms.

**Example 4.5.** If we consider the situations given in Example 4.3 then $\rho(X_1) = \rho(Y_1)$ and $\rho(X_N) - \rho(X_2) = 2 = \rho(Y_N) - \rho(Y_2)$, but $\phi^{E}_1(X_N) = 2 < 9 = \phi^{E}_1(Y_N)$. Moreover, $\phi^{E}_2(Y_N) = 7$, that is $\phi^{E}_1(Y_N) + \phi^{E}_2(Y_N) = 16 > 9 = \rho(Y_N)$, so the Euler method satisfies neither Equal Treatment Property, nor Strong Monotonicity, nor Core Compatibility. The reason of the last one is that in our example the measure of risk is not differentiable.

The properties of the Shapley method are described in Csóka and Pintér (2010), they show that it is the only method that is efficient and fulfills both Equal Treatment Property and Strong Monotonicity. Young (1985) on page 69 shows a counterexample, where the Nucleolus method does not fulfill Strong Monotonicity.

Summarizing our results we can say that the best performing methods are the game theoretic methods: the Shapley method and the Nucleolus method, as both of them satisfy two properties out of the examined three and we have counterexamples in which one requirement is not satisfied by them. The question is, how often those counterexamples happen for return distributions having empirically observable characteristics. To answer we simulate returns, first without, then with liquidity considerations.

### 4.2 Simulation results

From the three presented axioms Equal Treatment Property is crucial, it should be satisfied all the time. From a practical point of view it is interesting to know to what extent the other two requirements are violated for distributions having empirically observable characteristics. In this paper we concentrate on Core Compatibility, while we will examine Strong Monotonicity in a subsequent paper. Moreover, since the Shapley method is the only method that is efficient and fulfills both Equal Treatment Property and Strong Monotonicity, checking its average core compatibility for return distributions having empirically observable characteristics can give an answer to the level of impossibility of fair risk allocation in practice. Nevertheless using simulation we check Core Compatibility of all the mentioned methods,
except the Nucleolus method and the Euler method defined by Boonen et al. (2010), since we know that they are 100% core compatible.

In the simulation we model the returns of a financial unit (for instance a bank) consisting of first three, then four subunits (for instance divisions), assuming each has invested into assets worth $100 million. We generated return time series using random correlation matrices. As a first approach, we used normal distribution, then we also ran the simulation with Student t-distribution. From the simulated returns changes in invested values, from them risks of the coalitions (Expected Shortfall at 99%), the allocation methods and their average core compatibility can be calculated.

It was necessary to include the Student t-distribution, since in practice extremely low and high returns occur more often than it is implied by the normal distribution. That is, the tails of the distribution are heavier than of the normal distribution (Cont, 2001). This is called the stylized fact of heavy tails, first observed by Mandelbrot (1963) on cotton prices. The parameter of the t-distribution, the degrees of freedom (from now on $\nu$) shows how heavy the tails are. The lower the degrees of freedom the heavier the tails are. We ran the simulation using t-distributions with degrees of freedom 3 (following the “cubic law of returns”) and a lighter tail with $\nu=10$.

To get the return series we first generated correlation matrices (more precisely, first Cholesky matrices) and the standard deviations of the assets. For each correlation matrix we generated a lower triangular matrix consisting of elements uniformly distributed on the interval $(-1, 1)$. We multiplied this by its transpose and we normed the matrix - this way we got a correlation matrix. The standard deviations were generated from the interval $(0.01, 0.04)$, using uniform distribution. Then, we generated first three, later four, independent normally and t-distributed time series consisting of 1000 elements, that we multiplied by the Cholesky matrices and the standard deviations. This way finally we got multivariate normal and t-distributions. We generated Cholesky matrices 100000 times and for all Cholesky matrices a random return series consisting of 1000 elements. In the determination of the iteration number we considered both running time (7-15 minutes for each setting) and the stability of the results (the precision is around 0.3%). We chose to generate return series consisting of 1000 elements, because it equals four years of return data (assuming to have 250 trading days a year) that is considered to be enough for more or less exact risk assessment.

Our aim was, similarly to Homburg and Scherpereel (2008) to determine how many percent of the examined cases resulted Core Compatible allocation. Nevertheless, contrary to them, we chose a coherent measure of risk, Expected Shortfall, not Value at Risk and we did not only consider normal distribution, furthermore we examined more risk capital allocation methods.
The closer is the ratio of core compatible outcomes to 100% the better the method is, naturally.

Table 7 shows the rounded results for three players. We ran the simulations using different significance levels (95% and 99%) but as we could not observe a remarkable qualitative difference (Core Compatibility is changed at most with 4%), we will only publish the results for 99% here.

<table>
<thead>
<tr>
<th>Method/Distribution</th>
<th>Normal</th>
<th>Student-(t (\nu = 10))</th>
<th>Student-(t (\nu = 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Activity based</td>
<td>26.2%</td>
<td>26.5%</td>
<td>27.6%</td>
</tr>
<tr>
<td>Beta</td>
<td>80.8%</td>
<td>74.3%</td>
<td>57.8%</td>
</tr>
<tr>
<td>Incremental</td>
<td>20.3%</td>
<td>19.5%</td>
<td>19.4%</td>
</tr>
<tr>
<td>Cost gap</td>
<td>99.8%</td>
<td>99.7%</td>
<td>99.1%</td>
</tr>
<tr>
<td>Euler</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>Shapley</td>
<td>59.3%</td>
<td>57.1%</td>
<td>57.1%</td>
</tr>
<tr>
<td>Nucleolus</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Table 7: Average Core Compatibility in case of three subunits

As Table 7 shows that the Cost gap, the Euler and the Nucleolus methods performed much better than the other four methods. The Incremental and The Activity based methods showed relatively poor performance, but the Shapley and the Beta methods also obeyed the criterion of core compatibility in a large proportion of the cases.

Table 8 shows our simulation results for four players.

<table>
<thead>
<tr>
<th>Method/Distribution</th>
<th>Normal</th>
<th>Student-(t (\nu = 10))</th>
<th>Student-(t (\nu = 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Activity based</td>
<td>10.2%</td>
<td>10.7%</td>
<td>11.6%</td>
</tr>
<tr>
<td>Beta</td>
<td>73.5%</td>
<td>65.1%</td>
<td>42.9%</td>
</tr>
<tr>
<td>Incremental</td>
<td>6.7%</td>
<td>6.4%</td>
<td>6.1%</td>
</tr>
<tr>
<td>Cost gap</td>
<td>97.6%</td>
<td>97%</td>
<td>95.5%</td>
</tr>
<tr>
<td>Euler</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>Shapley</td>
<td>39.6%</td>
<td>38.6%</td>
<td>39.1%</td>
</tr>
<tr>
<td>Nucleolus</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Table 8: Average Core Compatibility in case of four subunits

Comparing the ratios of Core Compatible allocations with four players to our previous results with three players we can generally say that the ratio of core compatible allocations is weakly decreasing as we increase the number of players. The intuition can be that with more players the core itself gets smaller.
Let us consider the tendencies with respect to the fatness of the tails. Going from the normal and decreasing the degrees of freedom of the \( t \)-distribution we have fatter tails. The fatter the tails the lower the core compatibility of the methods tends to be, except for the activity based method and the Shapley method in case of four subunits. Thus the more extreme events usually generate more extreme claims of the coalitions, which are usually harder to satisfy.

The results of the simulation with four players outline the fact that if we are looking for a method that is suitable for practical application concentrating on Core Compatibility, the only possible candidates are the Cost gap, the Euler and the Nucleolus methods. The low ratio of the Shapley value shows that without liquidity considerations the impossibility of the fair risk allocation is a practical problem.

Our general conclusion is the following. Even if we are not mainly interested in Core Compatibility, we have only a few relevant possibilities left. Our analytic results in Table 3 show that only two methods satisfy more than one out of the three axioms: the Shapley and the Nucleolus methods. As the Shapley method is the only method that is efficient and fulfills both Equal Treatment Property and Strong Monotonicity, the low ratio of Shapley method’s Core Compatibility means that in practical applications we really have to give up one of the three properties.

References


Tasche D (2008) Capital allocation to business units and sub-portfolios: the Euler principle
