Shapley value for assignment games*

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Abstract

We consider the problem of the axiomatization of the Shapley value on the class of assignment games. We show that Shapley’s original \cite{21}, Young’s \cite{24}, Chun’s \cite{7}, van den Brink’s \cite{2}, (5-6) Hart and Mas-Colell’s \cite{12} potential function and consistency approaches and Roth’s \cite{19} characterization do not work on the class of assignment games. We also consider Myerson’s \cite{15} axiomatization of the Shapley value, and show that it is valid on the class of assignment games.

1 Introduction

The history of assignment games goes back to the XIX. century to Böhm-Bewerk’s \cite{1} horse market model. Later Shapley and Shubik \cite{23} introduced the formal, modern concept of assignment games.

One of the most popular solution concepts is the Shapley value (Shapley \cite{21}). Numerous axiomatizations of the Shapley value are known in the literature, in this paper we focus on the followings: (1) Shapley’s original \cite{21} also discussed by Dubey \cite{8}, Peleg and Sudhölter \cite{17}, (2) Young’s \cite{24} also discussed by Moulin \cite{14}, Pintér \cite{18}, (3) Chun’s \cite{7}, (4) van den Brink’s \cite{2}, (5-6) Hart and Mas-Colell’s \cite{12} potential function and consistency approaches, (7) Roth’s \cite{19} and (8) Myerson’s \cite{15} characterization.

First, we examine (1)-(7) characterizations of the Shapley value on the class of assignment games, and conclude that the class of assignment games

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is freaky from the viewpoint of the Shapley value axiomatizations: none of these characterizations is valid on the class of assignment games.

As a byproduct, we also show that Chun’s coalitional strategic equivalence, the marginality property \((EMP)\) and the strong monotonicity \((SM)\) axioms are not equivalent on the class of assignment games. Moreover, it turns out that on the class of assignment games the axioms anonymity \((AN)\) and symmetry (or equal treatment property \((ETP)\)) are not related to each other.

Second, we provide a positive result. Myerson [15] introduces for every TU-game and communication graph on a player set \(N\) a corresponding restricted game which is a TU-game in which the worth of coalitions is determined by the original game, taking account of the restrictions on the possibility of forming coalitions. The Myerson value then is the solution for (communication) graph games which assigns to every game and communication graph the Shapley value of this restricted game. Besides introducing his solution, Myerson [15] also characterizes it by the axioms of component efficiency and fairness, and shows that it is stable for superadditive graph games.

We argue that these results hold for any comprehensive subclass of graph games, in particular on the class of bipartite graph games. We refer to an arbitrary TU-game as a graph game on a particular graph if the restricted game on that graph equals the game itself. We argue that every assignment game is a graph game on the graph where all buyers are directly linked with all sellers. Consequently, Myerson’s [15] characterization of the Shapley value for graph games yields a characterization of the Shapley value on the class of assignment games and of the Shapley solution for assignment situations by two axioms referred to as submarket efficiency and valuation fairness.

Submarket efficiency states that the sum of the payoffs of all players in a submarket equals the worth of that submarket, where a submarket in an assignment situation is a set of buyers and sellers such that all buyers in this set have zero valuation for the goods offered by the sellers outside the set, and all buyers outside the set have zero valuations for the goods offered by sellers inside the set.

Valuation fairness states that changing the valuation of one particular buyer for the good offered by a particular seller changes the payoffs of this buyer and seller by the same amount. Further, the Shapley solution is stable in the sense that the payoffs of a buyer \(i\) and seller \(j\) do not decrease if we only increase the valuation of buyer \(i\) for the good offered by seller \(j\).

The setup of the paper is as follows. In section 2 we introduce the notations, notions used throughout the paper. Section 3 is on the (1)-(7) axiomatizations of the Shapley value on the class of assignment games. In Section 4
we characterize the Shapley solution for assignment situations. Some results on the relations of some axioms are relegated to the Appendix 5.

2 Definitions

Notation: $|N|$ is for the cardinality of set $N$, $\mathcal{P}(N)$ denotes the power set of $N$. Let $N$ be a non-empty, finite set, then $v : \mathcal{P}(N) \to \mathbb{R}$ such that $v(\emptyset) = 0$ is called transferable utility (TU) game with player set $N$. The class of transferable utility games with player set $N$ is denoted by $\mathcal{G}^N$.

It is well known that $\mathcal{G}^N$ is isomorphic with $\mathbb{R}^{2^{|N|}-1}$, therefore we regard $\mathcal{G}^N$ and $\mathbb{R}^{2^{|N|}-1}$ as identical. Moreover, $\forall v \in \mathcal{G}^N, \forall \beta \in \mathbb{R}^N: v \oplus \beta \in \mathcal{G}^N$ is defined as follows $\forall S \subseteq N: v \oplus \beta(S) \equiv v(S) + \sum_{i \in S} \beta_i$, where $\beta_i$ is component $i$ of vector $\beta$.

Furthermore, if $P$ is a matrix, then $A \subseteq P$ is for a subset of its elements.

For any $v \in \mathcal{G}^N$, $i \in N$ and $A \subseteq N$: let $v'_i(A) \equiv v(A \cup \{i\}) - v(A)$. Then $v'_i$ is called player $i$’s marginal contribution function in game $v$. Put it differently, $v'_i(A)$ is player $i$’s marginal contribution to coalition $A$ in game $v$.

Let $v \in \mathcal{G}^N$ be arbitrarily fixed. Players $i, j \in N$ are symmetric in game $v$, $i \sim v j$, if $\forall A \subseteq N$ such that $i, j \notin A: v'_i(A) = v'_j(A)$. Furthermore, if for some $i \in N$: $v'_i = 0$, then we say $i$ is a null player in game $v$. The set of null players in game $v$ is denoted by $NP(v)$.

Let $T \subseteq N$, $T \neq \emptyset$ be arbitrarily fixed, and $\forall A \subseteq N:

$$ u_T(A) = \begin{cases} 1, & \text{if } T \subseteq A \\ 0, & \text{otherwise} \end{cases} $$

Game $u_T$ is called unanimity game on coalition $T$.

Function $\psi : A \to \mathbb{R}^N$, where $A \subseteq \mathcal{G}^N$, is called solution on set $A$. In this paper we assume that a solution is single valued (more exactly: the range of the solution consists of singleton sets).

In order to discuss the axioms of a solution, we need the following notions: Let $v \in \mathcal{G}^N$ and $T \subseteq N$, $T \neq \emptyset$ be arbitrarily fixed. Then the subgame of $v$ on $T$, $v^T \in \mathcal{G}^T$, is defined as follows, $\forall S \subseteq T$:

$$ v^T(S) \equiv v(S) $$

It is clear that $v^T$ must be defined only on the subsets of $T$.

Let $A \subseteq \mathcal{G}^N \equiv \bigcup_{\emptyset \neq S \subseteq N} \mathcal{G}^S$ be an arbitrary class of games, and $\psi$ be an arbitrary solution on $A$. Moreover $\forall v \in A$, $\forall T \subseteq N$, $T \neq \emptyset$ such that $\forall S \subseteq T$, $S \neq \emptyset$: the subgame of $v$ on coalition $S \cup (N \setminus T)$, $v^{S \cup (N \setminus T)}$ is in $A$: let
\[ v_{T,\psi}(S) \triangleq v(S \cup (N \setminus T)) - \sum_{i \in N \setminus T} \psi_i(v_{S \cup (N \setminus T)}) , \]

and \( v_{T,\psi}(\emptyset) \triangleq 0 \). Then \( v_{T,\psi} \in \mathcal{G}^T \) is called the \( \psi \)-reduced game of \( v \) on coalition \( T \).

Next, we introduce some axioms of a solution:

Solution \( \psi \) on \( A \subseteq \mathcal{G}^N \) is / satisfies

- Pareto optimal (PO), if \( \forall v \in A: \sum_{i \in N} \psi_i(v) = v(N) \),
- null player property (NP), if \( \forall v \in A: (v_i' = 0) \Rightarrow (\psi_i(v) = 0) \),
- anonymous (AN), if \( \forall v \in A, \forall \pi \text{ permutation on } N \text{ such that } v \circ \pi \in A: \pi \circ \psi(v) = \psi(v \circ \pi) \),
- equal treatment property (ETP), if \( \forall v \in A: (i \sim^v j) \Rightarrow (\psi_i(v) = \psi_j(v)) \),
- covariant under strategic equivalence (COV), if \( \forall v \in A, \forall \alpha > 0, \forall \beta \in \mathbb{R}^N \text{ such that } \alpha v \oplus \beta \in A: \psi(\alpha v \oplus \beta) = \alpha \psi(v) + \beta \),
- additive (ADD), if \( \forall v, w \in A \) such that \( v + w \in A \): \( \psi(v + w) = \psi(v) + \psi(w) \),
- fairness property (FP), if \( \forall v, w \in A, \forall i, j \in N \text{ such that } v + w \in A \) and \( i \sim^w j \): \( \psi_i(v + w) - \psi_i(v) = \psi_j(v + w) - \psi_j(v) \),
- coalitional strategic equivalence (CSE), if \( \forall v \in A, \forall i \in N, \forall T \subseteq N, \forall \alpha > 0 : (i \in NP(w_T) \text{ and } v + \alpha u_T \in A) \Rightarrow (\psi_i(v) = \psi_i(v + \alpha u_T)) \),
- equal marginality property (EMP), if \( \forall v, w \in A, \forall i \in N: (v_i' = w_i') \Rightarrow (\psi_i(v) = \psi_i(w)) \),
- strong monotonicity (SM), if \( \forall v, w \in A, \forall i \in N: (v_i' \leq w_i') \Rightarrow (\psi_i(v) \leq \psi_i(w)) \).

Moreover, if solution \( \psi \) is defined on \( A \subseteq \Gamma^N \), then it is

- consistent (CON), if \( \forall T \subseteq N, \forall v \in A \cap \mathcal{G}^T, \forall S \subseteq T, S \neq \emptyset \text{ such that } v_{S,\psi} \in A: \forall i \in S: \psi_i(v_{S,\psi}) = \psi_i(v) \).
It is worth noticing that Chun’s original definition of CSE is different from ours. He defined CSE as "ψ is coaltional strategic equivalent (CSE), if \( \forall v \in A, \forall i \in N, \forall T \subseteq N, \forall \alpha \in \mathbb{R}: (i \in NP(u_T) and v + \alpha u_T \in A) \Rightarrow (\psi_i(v) = \psi_i(v + \alpha u_T))." However if for some \( \alpha < 0: v + \alpha u_T \in A, \) then by \( w \bowtie v + \alpha u_T \) and \( \beta \bowtie -\alpha \) we get to our definition "(i \in NP(u_T) and w + \beta u_T \in A) \Rightarrow (\psi_i(w) = \psi_i(w + \beta u_T))," where \( \beta > 0, \) i.e. the two definitions are equivalent.

The axioms EMP and Chun’s CSE are not equivalent, e.g. on the class of assignment games CSE is strictly weaker than EMP (and EMP is strictly weaker than SM, see Examples 5.2 and 5.3).

The two following lemmata give some relations among the above defined axioms. The proofs are obvious, so they are omitted.

**Lemma 2.1.** Let \( \psi \) be an arbitrary solution on set \( A \subseteq \mathcal{G}^N. \) Then

- if \( \psi \) is ETP and ADD, then it is FP,
- if \( \psi \) is SM, then it is EMP,
- if \( \psi \) is EMP, then it is CSE.

**Lemma 2.2.** Let \( \psi \) be an arbitrary solution on the class of games \( A \subseteq \mathcal{G}^N \) containing game \( [1] \). Then

- if \( \psi \) is NP and FP, then it is ETP,
- if \( \psi \) is EMP and \( \psi(0) = 0 \), then it is NP.

In this paper we focus on the Shapley solution (Shapley [21]): For any \( v \in \mathcal{G}^N \) and \( \forall i \in N: \) the Shapley value of player \( i \) in game \( v \)

\[
\phi_i(v) = \sum_{A \not= \emptyset} \frac{v'_i(A)}{|A|!(|N \setminus A| - 1)!}.
\]

Furthermore let \( \phi \) denote the Shapley solution.

The next proposition is well-known in the literature., hence we omit its proof.

**Proposition 2.3.** The Shapley solution is PO, NP, AN, ETP, CON, COV, ADD, FP, CSE, EMP and SM.

\(^1\)0 is for the zero game.
Now we are ready to focus on the class of assignment games (Shapley and Shubik [23]).

Let $B, S \subseteq N$ be non-empty disjoint sets such that $N = B \cup S$, and $A$ (its general element is $a_{ij}$) be a $|B| \times |S|$ non-negative matrix. Furthermore $\forall T \subseteq N$: let $P_A(T) \triangleq \{A \subseteq A \mid \forall a_{ij}, a_{i',j'} \in A : (i,j), (i',j') \in (B \cap T) \times (S \cap T)$ and $(i,j) = (i',j')$ or $\{i,j\} \cap \{i',j'\} = \emptyset\}$, and $\forall T \subseteq B \cup S$: let

$$v_a(T) \triangleq \max_{A \in P_A(T)} \sum_{a_{ij} \in A} a_{ij}.$$  

Then function $v_a$ is called assignment game with player sets $B, S$ for assignment situation $a$ with matrix $A$. Henceforth let $G^{B,S}$ be for the class of assignment games with player sets $B, S$.

Moreover, the elements of

$$\arg \max_{A \in P_A(T)} \sum_{a_{ij} \in A} a_{ij}$$

are called the maximal matchings of coalition $T$.

The assignment games can be interpreted as follows: $B$ and $S$ are for the sets of buyers and sellers respectively. Buyer $i$ and seller $j$ can make a deal in worth $a_{ij}$, buyer with buyer and seller with seller can make only worthless business. For any set of buyers and sellers $T$, the value of this coalition is the maximum aggregated worth of the deals the involved players can achieve contingent on every player can make business only with at most one other player.

Since in the definition of assignment games, $B$ and $S$ are non-empty, therefore in this paper every assignment game has at least two players. It is worth noting that if $u_T \in G^{B,S}$, then $|T| = 2$, and that $\forall v \in G^{B,S}, \forall \beta \in \mathbb{R}^{B+S}$: if $v \oplus \beta \in G^{B,S}$, then $\beta = 0$.

### 2.1 Communication graph games

Myerson [15] introduced a model in which it is assumed that the players in a game $v$ are part of a communication structure that is represented by an undirected graph $(N, L)$, with the player set $N$ as the set of nodes and $L \subseteq \{\{i,j\} | i,j \in N, i \neq j\}$ being a collection of edges, i.e. subsets of $N$ such that each element of $L$ contains precisely two elements of $N$. Since in this paper the nodes in a graph represent the players in a game we use the same notation for the set of nodes as the set of players, and refer to the nodes in a graph just as players. Further, because the elements of $L$ represent the

\footnote{We use the convention that the empty sum is 0.}
binary communication links between the players, we call them links instead of edges. Since we assume the set $N$ to be fixed we denote a graph on $N$ just by its set of links $L$ and refer to this as the graph if this does not lead to confusion. We denote the class of all possible sets of links on $N$ by $L^N$.

A sequence of $k$ different nodes $(i_1, \ldots, i_k)$ is a path between players $i_1$ and $i_k$ in $L \in L^N$ if $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k - 1$. A coalition $S \subseteq N$ is connected in graph $L$ if every pair of players in $S$ is connected by a path that only contains players from $S$, i.e. for every $i, j \in S$, $i \neq j$, there is a path $(i_1, \ldots, i_k)$ such that $i_1 = i$, $i_k = j$ and $\{i_1, \ldots, i_k\} \subseteq S$. Coalition $T \subseteq S$ is a component of $S$ in graph $L$ if it is a maximally connected subset of $S$, i.e. $T$ is connected in $L(S)$ and for every $h \in S \setminus T$ the coalition $T \cup \{h\}$ is not connected in $L(S)$, where $L(S) = \{\{i, j\} \in L | \{i, j\} \subseteq S\}$. We denote the set of components of $S \subseteq N$ in $L$ by $C_L(S)$.

A pair $(v, L) \in \mathcal{G}^N \times L^N$ is referred to as a graph game on $N$. In the graph game $(v, L)$ players can cooperate if and only if they are able to communicate with each other, i.e. a coalition $S$ can realize its worth $v(S)$ if and only if $S$ is connected in $L$. Whenever this is not the case, players in $S$ can only realize the sum of the worths of the components of $S$ in $L$. As introduced by Myerson [15], this yields the restricted game $v^L \in \mathcal{G}^N$ given by

$$v^L(S) = \sum_{T \in C_L(S)} v(T) \quad S \subseteq N. \quad (1)$$

The Myerson value is the solution $\mu: \mathcal{G}^N \times L^N \to \mathbb{R}^N$ that is obtained by taking the Shapley value of the restricted game $v^L$, i.e.

$$\mu(v, L) = \phi(v^L), \text{ for all } v \in \mathcal{G}^N \text{ and } L \in L^N.$$ 

Besides introducing this value, Myerson [15] gives a characterization by the axioms of component efficiency and fairness. Solution $\psi$ satisfies

- Component efficiency, if for every graph game $(v, L) \in \mathcal{G}^N \times L^N$ and every component $S \in C_L(N)$, it holds that $\sum_{i \in S} f_i(v, L) = v(S)$,
- Fairness, if for every graph game $(v, L) \in \mathcal{G}^N \times L^N$ and every pair of players $i, j \in N$, it holds that $f_i(v, L) - f_i(v, L \setminus \{i, j\}) = f_j(v, L) - f_j(v, L \setminus \{i, j\})$.

Component efficiency states that the sum of the payoffs of all players in a component equals the worth of that component. Fairness states that deleting the link between two players changes their payoffs by the same amount. Moreover, Myerson [15] shows that his solution is stable in the sense that for superadditive games adding a link never hurts the two players incident with
that link. Game \( v \) is superadditive if \( v(S \cup T) \geq v(S) + v(T) \) for all \( S, T \subseteq N \) with \( S \cap T = \emptyset \).

**Theorem 2.4.** (Myerson [15])

(i) The Myerson value is the unique solution \( f: G^N \times \mathcal{L}^N \to \mathbb{R}^N \) that satisfies component efficiency and fairness.

(ii) For every graph game \((v,L) \in G^N \times \mathcal{L}^N\) with \( v \) superadditive, it holds that \( \mu_i(v, L) \geq \mu_i(v, L \setminus \{l\}) \) for every \( i \in l \in L \).

### 3 Axiomatizations of the Shapley value on the class of assignment games

In this section we consider two basic types of axiomatizations of the Shapley value: the first where the player set is not fixed, and the second where it is fixed, therefore we split this section into two subsections.

#### 3.1 Hart and Mas-Colell’s approaches

In this subsection we consider Hart and Mas-Colell’s [12] two approaches.

**Definition 3.1.** Let \( A \subseteq \Gamma^N = \bigcup_{T \subseteq N} \mathcal{G}^T \), \( P: A \to \mathbb{R} \) be a function, and \( \forall T \subseteq N, T \neq \emptyset, \forall v \in \mathcal{G}^T \cap A, \forall i \in T \) such that \( |T| = 1 \) or \( v^{T \setminus \{i\}} \in A \):

\[
P'_i(v) \triangleq \begin{cases} P(v), & \text{if } |T| = 1 \\ P(v) - P(v^{T \setminus \{i\}}) & \text{otherwise} \end{cases}
\]  

Furthermore if \( \forall v \in \mathcal{G}^T \cap A \) such that either \( |T| = 1 \) or \( \forall i \in T: v^{T \setminus \{i\}} \in A \):

\[
\sum_{i \in T} P'_i(v) = v(T) ,
\]

then \( P \) is called potential on \( A \).

**Definition 3.2.** \( A \subseteq \Gamma^N \) is subgame closed, if \( \forall T \subseteq N \) such that \( |T| > 1 \), \( \forall v \in \mathcal{G}^T \cap A \) and \( \forall i \in T: v^{T \setminus \{i\}} \in A \).

Since there is no game without players, in the above definition we require that subgame \( v^{T \setminus \{i\}} \) be in the set under consideration only if there are at least two players in \( T \).

**Proposition 3.3.** Let \( A \subseteq \Gamma^N \) be a subgame closed set of games. Then function \( P \) on \( A \) is a potential, if and only if \( \forall T \subseteq N, T \neq \emptyset, \forall v \in \mathcal{G}^T \cap A \) and \( \forall i \in T: P'_i(v) = \phi_i(v) \).
Proof. It comes directly from e.g. Peleg and Sudhölter [17] Theorem 8.4.4. (pp. 216-217). Q.E.D.

By now we are ready to look into the case of assignment games.

**Corollary 3.4.** On the class of assignment games there is a potential $P$ such that $\exists v \in \Gamma B \cup S$ assignment game and $\exists i$ player of $v$: $P_i'(v) \neq \phi_i(v)$.

Proof. Let $B \doteq \{i\}, S \doteq \{j\}, v \in G^{B,S}$ be an arbitrary assignment game. In this case, neither $v'(B \cup S) \setminus \{i\}$ nor $v'(B \cup S) \setminus \{j\}$ are assignment games.

In general, the potential is not well defined on the class of assignment games with two players, therefore one can give any value to these games. Since potential is defined recursively (see Definition 3.1), therefore its value on any game is determined by these arbitrarily fixed values. Summing up, there are as many as continuum different potentials on the class of assignment games. Q.E.D.

**Remark 3.5.** In the proof above we emphasized that the potential is not well defined on the class of assignment games with two players. Unfortunately, the situation is worse, e.g. on assignment games with only one seller or with only one buyer the potential is also not well defined.

However, if we define the assignment games as that the buyers’ and the sellers’ sets can be empty sets then, the not-well-definedness disappears, therefore, Hart and Mas-Colell’s potential function characterization becomes valid on this redefined class of assignment games.

Next we demonstrate that Hart and Mas-Colell’s approach based on the axiom consistency ($PO$, $COV$, $ETP$ and $CON$) does not work on the class of assignment games either.

**Example 3.6** ($\exists \psi$ solution on $\Gamma B \cup S \doteq \bigcup_{T \subseteq B \cup S, T \cap B \cup S \neq \emptyset} G^{T \cap B \cup S}$ such that it is $PO$, $AN$, $ETP$, $COV$, $CON$, but $\psi \neq \phi$). $\forall v \in G^{T \cap B \cup S} \subseteq \Gamma B \cup S$, $\forall i \in B \cup S$: let

$$\psi_i(v) \doteq \begin{cases} \frac{v(T)}{|T \setminus NP(v)|}, & \text{if } i \notin NP(v) \\ 0, & \text{otherwise} \end{cases}$$

It is worth noticing that for any assignment game $v \neq 0$, and for any player $i$: if $i \notin NP(v)$, then $\psi_i(v) > 0$.

Next we list the properties of $\psi$:  

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• $\psi$ is *PO, AN, ETP* and *COV*: It is left for the reader (if $v \oplus \beta \in G_{B, S}^B$, then $\beta = 0$).

• $\psi$ is *CON*: If $|B \cup S| = 2$ then $\forall v \in G_{B, S}^B$: $\psi(v) = \phi(v)$. Let $|B \cup S| > 2$, $v \in G_{B, S}^B$ and $T \subseteq B \cup S$ be arbitrarily fixed. We consider $v_{T, \psi}$. If $v_{T, \psi}$ is an assignment game then $\forall i \in T, \forall j \in (B \cup S) \setminus T$: $v_i \left( \{i, j\} \right) = 0$ if and only if $i \in NP(v)$, so $i \in NP(v^{(i) \cup ((B \cup S) \setminus T)})$, and $v_{T, \psi}(T) = \sum_{i \in T} \psi_i(v)$. Therefore, $i \in NP(v_{T, \psi})$ if and only if $i \in NP(v)$. Then $\forall i \in T$: $\psi_i(v_{T, \psi}) = \psi_i(v)$.

• $\psi \neq \phi$: It is left for the reader.

### 3.2 Further axiomatizations

In this subsection we present a slight, but positive result and a counterexample. First see the "trivial case."

**Theorem 3.7.** If $|B \cup S| = 2$, then solution $\psi$ on $G_{B, S}^B$ is *PO* and *ETP* if and only if $\psi = \phi$.

Henceforth we assume that $|B \cup S| > 2$, i.e. the class of assignment games under consideration is not the "trivial" one.

Next we introduce two axioms, which are specially interesting for assignment games (see Section 4).

Applied to assignment situations, component efficiency of a solution implies that the sum of the payoffs of all players in a submarket equals the worth of that submarket, where a submarket in assignment situation $a \in \mathcal{A}_{B, S}^B$ is a pair of sets $(B', S')$ with $B' \subseteq B$ and $S' \subseteq S$ such that $a_{i, j} = 0$ for all pairs $(i, j) \in (B' \times (S \setminus S')) \cup ((B \setminus B') \times S')$, i.e. a submarket is a set of buyers and sellers such that all buyers in this set have zero valuation for the goods offered by the sellers outside the set, and all buyers outside the set have zero valuation for the goods offered by sellers inside the set.

**Definition 3.8.** Let $v \in G_{B, S}^B$ be assignment game. Then $B', S'$ is a submarket of game $v$, if $B' \subseteq B$, $S' \subseteq S$, $B', S' \neq \emptyset$, and $\forall (i, j) \in (B' \times (S \setminus S')) \cup ((B \setminus B') \times S')$: $v_i \left( \{i, j\} \right) = 0$.

Pareto optimality (Efficiency) can be required for not only the whole market, but for the submarkets too. This property is called submarket efficiency.

Fairness of a solution applied to assignment situations implies that decreasing the valuation of one particular buyer for the good offered by a particular seller to zero, changes the payoffs of this buyer and seller by the same amount.
Definition 3.9. Solution \( \psi \) defined on \( G_{B,S} \) satisfies

- **submarket efficiency (SME)**, if \( \forall v \in G_{B,S}, \forall (B', S') \) submarket of \( v \):
  \[ \sum_{i \in B' \cup S'} \psi_i(v) = v(B' \cup S'), \]

- **valuation fairness property (VFP)**, if \( \forall v, w \in G_{B,S}, \forall i \in B, \forall j \in S, w(\{i, j\}) = 0, \forall \{k, l\} \in \mathcal{P}(B \cup S) \setminus \{i, j\}, v(\{k, l\}) = w(\{k, l\}): \)
  \[ \psi_i(v) - \psi_i(w) = \psi_j(v) - \psi_j(w). \]

It is easy to verify that SME implies PO.

Example 3.10 (\( \exists \psi \) solution on \( G_{B,S} \) such that it is SME, AN, ETP, COV, ADD, SM, but it is not VFP and \( \psi \neq \phi \)). Let

\[ OR_B \triangleq \{ \tau \in OR(B \cup S) \mid (\tau(i) \leq |B|, i \in B \cup S) \Rightarrow (i \in B) \}, \]

(the buyers come first) and

\[ OR_S \triangleq \{ \tau \in OR(B \cup S) \mid (\tau(i) \leq |S|, i \in B \cup S) \Rightarrow (i \in S) \}, \]

(the sellers come first) where \( OR(B \cup S) \) is for the set of all (linear) orderings on set \( B \cup S \).

Furthermore \( \forall i \in B \cup S, \forall v \in G_{B,S} \): let

\[ \psi^B_i(v) \triangleq \frac{1}{|OR_B|} \left( \sum_{\tau \in OR_B} (v(\{j \in B \cup S \mid \tau(j) \leq \tau(i)\}) \right. \]
\[ \left. -v(\{j \in B \cup S \mid \tau(j) < \tau(i)\}) \right), \]

and

\[ \psi^S_i(v) \triangleq \frac{1}{|OR_S|} \left( \sum_{\tau \in OR_S} (v(\{j \in B \cup S \mid \tau(j) \leq \tau(i)\}) \right. \]
\[ \left. -v(\{j \in B \cup S \mid \tau(j) < \tau(i)\}) \right). \]

Then \( \forall i \in B \cup S, \forall v \in G_{B,S} \): let

\[ \psi_i(v) \triangleq \frac{\psi^B_i(v) + \psi^S_i(v)}{2}. \]

Next we list the properties of \( \psi \):
• \( \psi \) is a convex combination of random order values: It is left for the reader.

• \( \psi \) is SME, AN, COV, ADD and SM: It is left for the reader.

• \( \psi \) is ETP: If in game \( v i \sim^v j \), then there can be two cases: (1) either both players are buyers or both are sellers. In this case AN implies ETP.

\[
\forall k \in (B \cup S) \setminus \{i, j\}: v(\{i, k\}) = v(\{j, k\}) = 0. \text{W.l.o.g. we can assume that } i \in B \text{ and } j \in S. \text{Then } \psi^B_i(v) = 0 \text{ and } \psi^B_j(v) = v(\{i, j\}), \text{similarly, } \psi^S_i(v) = v(\{i, j\}) \text{ and } \psi^S_j(v) = 0. \text{Therefore } \psi_i(v) = \psi_j(v).
\]

• \( \psi \) is not VFP: It is left for the reader.

• \( \psi \neq \phi \): It is left for the reader.

In conclusion:

**Corollary 3.11.** The following points hold:

1. Hart and Mas-Colell’s potential function characterization of the Shapley value on the class of assignment games is not valid,

2. Hart and Mas-Colell’s axiomatization (PO, ETP, COV, CON) of the Shapley value on the class of assignment games is not valid,

3. Shapley’s axiomatization (PO, NP, ETP (AN), ADD) of the Shapley value on the class of assignment games is not valid,

4. van den Brink’s axiomatization (PO, NP, FP) of the Shapley value on the class of assignment games is not valid,

5. Chun’s axiomatization (PO, ETP, CSE) of the Shapley value on the class of assignment games is not valid,

6. Young’s axiomatization (PO, ETP, SM) of the Shapley value on the class of assignment games is not valid.

**Proof.** See Lemma 2.1, Corollary 3.4 and Examples 3.6, 3.10. Q.E.D.

**Remark 3.12.** Roth [19] showed that the Shapley value can be interpreted as a von Neumann-Morgenstern utility function. His set up can be adapted to the class of assignment games as well. However, since his result is closely connected to axioms PO, NP, AN and ADD i.e. to Shapley’s original axiomatization, therefore neither his characterization is valid on the class of non-trivial assignment games.
4 An axiomatization of the Shapley solution for assignment situations

Given an assignment situation with buyer-seller partition \((B, S)\), consider the communication graph on \(N\) in which the links reflect all matching possibilities. So, the graph on \(N\) is the “full” bipartite graph \(L^{B,S} = \{\{i, j\} \subset N \mid i \in B, j \in S\}\). Since every coalition that contains at least one seller and at least one buyer is connected in \(L^{B,S}\), and all coalitions that contain only buyers or only sellers have worth zero, for every graph restricted assignment game \((v_a, L^{B,S})\), \(a \in A^{B \times S}\), it holds that the Myerson restricted game \((v_a)_{L^{B,S}}\) is equal to \(v_a\). Thus, an assignment game \(v_a\) is a graph game on the corresponding full bipartite graph.

A general bipartite graph (i.e. that is not necessarily full) on \(B \times S\) is a graph \(L \subseteq L^{B,S}\) with \(\{i, j\} \in L\) only if \(i \in B\) and \(j \in S\). We denote the class of all bipartite graphs between the sets of buyers and sellers \(S\) by \(L^{B,S}\). Clearly, a matching is a bipartite graph that is not full. Note that if for an assignment game \(v \in G^{B,S}\) it holds that \(v = v^L\) for some bipartite graph \(L \in L^{B,S}\), then also \(v_a = (v_a)^{L'}\) for every bipartite graph \(L' \supseteq L\). The minimal bipartite graph \(L_m^v\) such that \(v_a = (v_a)^{L_m^v}\) is \(L_m^v = \{\{i, j\} \subset N \mid i \in B, j \in S\}\) and \(a_{i,j} > 0\).

Although not explicitly written, Theorem 2.4.(ii) holds more general in the sense that component efficiency and fairness characterize the Myerson value on any restricted class of graph games \(G_N \times C\) such that \(C \subseteq L^N\) is comprehensive, i.e. for any \(L \in \mathcal{C}\) and \(L' \subseteq L\) it holds that \(L' \in \mathcal{C}\). For example, the class \(L^{B,S}\) of all bipartite graphs between the sets of buyers \(B\) and sellers \(S\) satisfies this property.

Next, similar to the proof of Theorem 2.4(i) (see Myerson [15]) the following can be shown.

**Theorem 4.1.** The Shapley solution \(\phi\) is the unique solution for assignment situations that satisfies submarket efficiency (SME) and valuation fairness (VFP).

**Proof.** We first prove that the Shapley solution satisfies the two axioms.

(i) Note that \((B', S')\) being a submarket in the assignment situation \(a \in A^{B,S}\) implies that \(C = B' \cup S'\) is a game-component in \(v_a\), i.e. \(v_a(E) = v_a(E \cap C) + v_a(E \setminus C)\) for all \(E \subseteq B \cup S\). The Shapley solution satisfying SME for assignment situations then follows from the Shapley value satisfying Component Efficiency (see Subsection 2.1), i.e. \(\sum_{i \in C} \phi_i(v) = v(C)\) for every game-component \(C\) in any game \(v \in G^N\), see Chang and Kan [5].

(ii) If \(a_{g,h} = a_{g,h}\) for all \(g \in B \setminus \{j\}, h \in S \setminus \{i\}\), then all coalitions which worth in \(v_a\) is different from its worth in \(v_\pi\) should contain both players \(i\) and...
valuation of one particular buyer for the good offered by a particular seller
an even stronger valuation fairness property which states that changing the
uniquely determined. Q.E.D.

Since this can be done for all submarkets, the payoffs
for all submarkets, and thus $f(v)$ is determined (uniquely) by
SME.

We prove uniqueness\footnote{This goes along similar lines as Myerson \cite{15} proves uniqueness of the Myerson value for graph games by induction on the number of links $|L|$.} by induction on the number of non-zero valuations
$m(a) = |M(a)|$, where $M(a) = \{(i, j) \in B \times S \mid a_{i,j} > 0\}$. Suppose that
$f : \mathcal{G}^{B,S} \to \mathbb{R}^N$ satisfies SME and VFP. If $m(a) = 0$, then all singleton
player sets form a submarket, and thus $f(v)$ is determined (uniquely) by
SME.

Proceeding by induction, assume that $f(a')$ is (uniquely) determined
whenever $0 \leq m(a') < m(a)$. Take any submarket $C = B' \cup S'$, $B' \subseteq B$, $S' \subseteq S$. If $|C| = 2$ ($C$ cannot be 1), then $f_i(a)$, $i \in C$, is determined
(uniquely) by SME and VFP.

Otherwise, if $|C| > 2$, then $C$ is connected in $L^m_a$, let $c = |C|$. Take
$L' \subseteq L^m_a$ such that the restriction of $L'$ to $C$ is a connected tree, this means
that $C$ is connected in $L'(C)$ and $|L'(C)| = c - 1$. Assume without loss of
generality that the players in $C$ are labeled $1, \ldots, c$ such that \{i, i + 1\} $\in L'$
for all $i \in \{1, \ldots, c - 1\}$. Further, assume without loss of generality that
1 $\in B$ (and thus $i \in B$ and $i + 1 \in S$ if $i \leq c - 1$ is odd).

Next, define the following assignment situations $a^i \in \mathcal{A}^{B,S}$, $i \in C$. For
$i \in \{1, \ldots, c - 1\}$ odd, define $a^i_{1,\ldots,i+1} = 0$, and $a^i_{g,h} = a_{g,h}$ for $g \in B \setminus \{i\}$,
h $\in S \setminus \{i + 1\}$. For $i \in \{2, \ldots, c - 1\}$ even, define $a^i_1,\ldots,i = 0$, and $a^i_{g,h} = a_{g,h}$
for $g \in B \setminus \{i - 1\}$, $h \in S \setminus \{i\}$.

From VFP it follows that

$$f_i(a) - f_i(a') = f_{i+1}(a) - f_{i+1}(a') \text{ for all } i \in \{1, \ldots, c - 1\} . \quad (3)$$

Since the values $f_i(a')$ and $f_{i+1}(a')$ for all $i \in \{1, \ldots, c - 1\}$ are determined
by the induction hypothesis, the system \cite{3} yields $c - 1$ linearly independent
equations in the $c$ unknown payoffs $f_i(a)$, $i \in C$. Adding the $c$th equation
$\sum_{i \in C} f_i(a) = v_a(C)$ which follows from SME, these payoffs are uniquely
determined. Since this can be done for all submarkets, the payoffs $f(a)$ are
uniquely determined. Q.E.D.

Note that from the proof it follows that the Shapley solution satisfies
an even stronger valuation fairness property which states that changing the
valuation of one particular buyer for the good offered by a particular seller
Theorem 4.2. Consider assignment situations a, π ∈ A^{B,S} such that for some i ∈ B, j ∈ S it holds that π_{i,j} ≥ a_{i,j}, and a_{g,h} = π_{g,h} for all g ∈ B \ {i}, h ∈ S \ {j}. Then φ_i(v_π) ≥ φ_i(v_a) and φ_j(v_π) ≥ φ_j(v_a).

Proof. This theorem follows straightforward from the marginal contributions of the players in the corresponding assignment games. For every a ∈ A^{B,S}, i ∈ B ∪ S and T ⊆ B ∪ S with i ∈ T define m_a^i(T) = v_a(T) − v_a(T \ {i}). Take assignment situations a, π ∈ A^{B,S} such that for some i ∈ B, j ∈ S it holds that π_{i,j} ≥ a_{i,j}, and a_{g,h} = π_{g,h} for all g ∈ B \ {i}, h ∈ S \ {j}.

If j ∉ T, then v_π(T) = v_a(T) and v_π(T \ {i}) = v_a(T \ {i}), and thus m_a^i(T) = m_a^i(T). On the other hand, if j ∈ T, then v_π(T) ≥ v_a(T) and v_π(T \ {i}) = v_a(T \ {i}), and thus m_a^i(T) ≥ m_a^i(T). Thus, m_a^i(T) ≥ m_a^i(T) for all T ⊆ B ∪ S, i ∈ T, implying that φ_i(v_π) ≥ φ_i(v_a). Similar it follows that φ_j(v_π) ≥ φ_j(v_a).

5 Appendix on the axioms

First we look into the relation of axioms AN and ETP. At first glance some could think that AN implies ETP, of course, for example on G^N it is true. However, in general, or particularly on the class of assignment games it is not true at all.

Let v ∈ G^{B,S} and permutation π on B ∪ S be arbitrarily fixed. Then v ◦ π ∈ G^{B,S} if and only if ∀i ∈ B: π(i) ∈ B and ∀i ∈ S: π(i) ∈ S. Therefore, while a buyer and a seller can be related to each other by ETP, it cannot happen by AN, i.e. on G^{B,S} AN does not imply ETP. Moreover, it is clear that ETP does not imply AN. In conclusion, in general AN and ETP are not related to each other.

E.g. if we substitute AN for ETP in Theorem 3.7 then that is not true more. Since on the class G^{B,S}, where |B| = |S| = 1 the axiom AN is meaningless, therefore if a solution is PO and AN then it is practically PO only. However there are as many as continuum PO solutions on the class of games under consideration.
Next we examine the relations of axioms CSE, EMP and SM. To do so we need the following lemma.

**Lemma 5.1.** Let \( v, w \in \mathcal{G}_{B,S} \) be arbitrarily fixed, and \( i \in B \ (i \in S) \) be such that \( \forall A \subseteq B \cup S, i \notin A, |A| = 2, A \cap S \neq \emptyset \ (A \cap B \neq \emptyset) \): \( v_i(A) > 0 \). Then \( v_i' = w_i' \) if and only if \( v = w \).

**Proof.** if: It is left for the reader.

only if: W.l.o.g. we can assume that \( i \in B \). Then if \( A \subseteq B \) or \( A \subseteq S \), \( |A| = 2 \), then \( v(A) = 0 = w(A) \). Moreover, if \( v_i' = w_i' \), then \( \forall A \subseteq B \cup S, |A| = 2, A \cap S \neq \emptyset, i \in A \): \( v(A) = w(A) \). Let \( k \in B \cup S, k \neq i, l \in S \) be arbitrarily fixed. Then \( v_i'({k, l}) > 0 \), hence \( v_i'({k, l}) = \max\{v({i, k}), v({i, l})\} - v({k, l}) \). However, we know that \( w({i, k}) = v({i, k}) \) and \( w({i, l}) = v({i, l}) \), therefore \( w({k, l}) = v({k, l}) \). Q.E.D.

The following example is on that axioms EMP and SM are not equivalent on the class of assignment games.

**Example 5.2** \((\exists \psi \text{ solution on } \mathcal{G}_{B,S} \ (|B \cup S| > 2) \text{ such that it is } PO, \ AN, \ ETP, \ EMP, \text{ but not } SM)\). \( \forall i, j \in B \cup S \) let

\[
E_{i,j} \triangleq \{ v \in \mathcal{G}_{B,S} \mid 0 < v({i, j}), \forall A \subseteq B \cup S, A \not\subseteq B, A \not\subseteq S, |A| = 2, A \cap {i, j} \neq \emptyset, A \neq {i, j} : 0 < v(A) < v({i, j}) \},
\]

and for any other \( A \subseteq B \cup S, |A| = 2 : v(A) = 0 \).

It is easy to verify that the sets \( E_{i,j} \) are disjoint. Furthermore, let \( E \triangleq \bigcup_{i,j \in B \cup S} E_{i,j}, \forall \psi \in \mathcal{G}_{B,S}, \forall i \in B \cup S : \psi_i(v) \triangleq \begin{cases} \frac{\phi_j(v) + \phi_j(v)}{2}, & \text{if } \exists j \in B \cup S : v \in E_{i,j} \\ \phi_i(v), & \text{otherwise} \end{cases} \).

Next we list the properties of \( \psi \):

- \( \psi \) is \( PO, \ AN \) and \( ETP \): It is left for the reader.
- \( \psi \) is \( EMP \): See Lemma 5.1
- \( \psi \) is not \( SM \): Let \( w \in E_{i,j} \) be arbitrary fixed. W.l.o.g. we can assume that \( |B| \geq 2 \) and \( i \in S \). Moreover, let \( v \in E_{i,j} \) be such that \( \forall A \subseteq B \cup S, A \not\subseteq S, |A| = 2, i \in A, A \neq {i, j} : v(A) \triangleq w(A) - \varepsilon, \) where \( \varepsilon > 0, \) and \( \forall A \subseteq B \cup S, |A| = 2, i \notin A \) or \( A \subseteq S \) or \( A = \{i, j\} : v(A) \triangleq w(A) \). Then \( v_i' \leq w_i' \) but \( \psi_i(v) > \psi_i(w) \ (\forall k \in (B \cup S) \setminus \{i, j\} : \psi_k(v) = \phi_k(v) \leq \phi_k(w) = \psi_k(w)) \).
The next example is on that CSE is not equivalent with EMP on the class of assignment games.

**Example 5.3** (∃ψ solution on \(G^B,S\) (\(|B \cup S| \geq 4\)) such that it is PO, AN, ETP, CSE but not EMP.) We consider two cases.

(1) Assume that \(|B|, |S| \geq 2\) and ∀\(i, j \in B \cup S\): let

\[
E_{i,j} = \{v \in G^B,S \mid \forall k, l \in B \cup S, \{k, l\} \neq \{i, j\} : v(\{k, l\}) < v(\{i, j\})\}.
\]

It is easy to verify that the sets \(E_{i,j}\) are disjoint, moreover let \(E = \bigcup_{i,j \in B \cup S} E_{i,j}\), and ∀\(v \in G^B,S\), ∀\(i \in B \cup S\):

\[
\psi_i(v) = \begin{cases} 
\frac{\phi_i(v) + \phi_j(v)}{2}, & \text{if } \exists j \in B \cup S : v \in E_{i,j} \\
\phi_i(v), & \text{otherwise}
\end{cases}
\]

Next we list the properties of \(\psi\):

- \(\psi\) is PO, AN and ETP: It is left for the reader.

- \(\psi\) is CSE: If \(v + \alpha w_T\), \(\alpha > 0\), is an assignment game, then (1) \(v + \alpha w_T \in E_T\), and (2) \(\forall A \subseteq B \cup S\), \(|A| = 2, A \neq T\): \(v \notin E_A\), i.e. \(v \notin E\) or \(v \in E_T\).

Therefore ∀\(k \in (B \cup S) \setminus T\):

\[
\psi_k(v) = \phi_k(v) = \phi_k(v + \alpha w_T) = \psi_k(v + \alpha w_T).
\]

- \(\psi\) is not EMP: Let \(i, k \in B\) and \(j, l \in S\), \(i \neq k\) and \(j \neq l\) be arbitrarily fixed. Furthermore, let \(v \in B \cup S\) be such that \(v(\{i, j\}) = 2, v(\{j, k\}) = 1, v(\{k, l\}) = 1\) and ∀\(i', j' \in B \cup S\), \(\{i', j'\} \notin \{\{i, j\}, \{j, k\}, \{k, l\}\} : v(\{i', j'\}) = 0\). Then \(v \in E_{i,j}\) and \(\psi_i(v) = 1\).

Furthermore, let \(w \in B \cup S\) be such that \(w(\{i, j\}) = 2, w(\{j, k\}) = 1, w(\{k, l\}) = 5\) and ∀\(i', j' \in B \cup S\), \(\{i', j'\} \notin \{\{i, j\}, \{j, k\}, \{k, l\}\} : w(\{i', j'\}) = 0\). Then \(w \notin E_{i,j}\), \(v'_i = 1\) and \(\psi_i(w) = \frac{11}{12}\).

(2) Assume that \(|B| = 1\) and \(|S| \geq 3\) (\(|S| = 1\) and \(|B| \geq 3\)). Let \(i \in B\) (\(i \in S\)), and \(v \in G^B,S\) be such an assignment game that \(v(\{i, j\}) = v(\{i, k\}) = 2\) and \(v(\{i, l\}) = 1\), where \(j, k, l \in B \cup S\) (different players), and \((B \cup S) \setminus \{i, j, k, l\} \subseteq NP(v)\) (the other players are null-players).
Moreover, let

\[ E_{i,j} = \{ w \in G^{B,S} \mid \exists \alpha \in \mathbb{R}_+ : w = v + \alpha u_{\{i,j\}} \} \]

\[ E_{i,k} = \{ w \in G^{B,S} \mid \exists \alpha \in \mathbb{R}_+ : w = v + \alpha u_{\{i,k\}} \} . \]

Then \( \forall w \in G^{B,S}, \forall h \in B \cup S: \) let

\[ \psi_h(w) = \begin{cases} 
0, & \text{if } w \in E_{i,j} \cup E_{i,k} \text{ and } h \neq i \\
\phi_h(w), & \text{if } w \in E_{i,j} \cup E_{i,k} \text{ and } h = i \\
\phi_h(w), & \text{otherwise}
\end{cases} \]

Next we list the properties of \( \psi: \)

- \( \psi \) is \( PO, AN, ETP \) and \( CSE \): It is left for the reader.
- \( \psi \) is not \( EMP \): Let \( w \in G^{B,S} \) such that such an assignment game that \( w(\{i,j\}) = w(\{i,k\}) = 3 \) and \( v(\{i,l\}) = 1 \), and \( (B \cup S) \setminus \{i,j,k,l\} \subseteq NP(v) \) (the other players are null-players). Then \( v'_i = v'_l, w \notin E_{i,j} \cup E_{i,k} \) and \( \psi_l(v) = 0 < \psi_l(w) = \psi_l(w). \)

Because of didactical reasons, in the above example we have presented two examples. It is easy to see that, however, it is possible to put the two examples together and to get only one "universal example."

In the above example we have assumed that \( |B \cup S| \geq 4 \). We have a reason for doing so.

**Remark 5.4.** It is a slight calculation to verify the following claim: If \( |B \cup S| \leq 3 \), then for any solution \( \psi \) on any \( A \subseteq G^{B,S} \): \( \psi \) is \( CSE \) if and only if it is \( EMP \).

W.l.o.g. we can assume that \( B = \{i\} \). If \( v, w \in G^{B,S} \) such that \( v'_i = w'_i \), then \( v = w \). If \( j \in S \) and \( v'_j = w'_j \), then two cases might happen. (1) \( k \in S \setminus \{j\} \): \( v(\{i,k\}) < v(\{i,j\}) \), then Lemma \[5.1\] implies that \( v = w \). (2) \( k \in S \setminus \{j\} \): \( v(\{i,j\}) \leq v(\{i,k\}) \), then \( \exists \alpha, \beta \in \mathbb{R}_+ \) such that \( v = w' + \alpha u_{\{i,j\}} \) and \( w = w' + \beta u_{\{i,k\}} \), where \( w' \in G^{B,S} \) such that \( w'({\{i,j\}}) = v(\{i,j\}) = w(\{i,j\}) \), \( w'({i,k}) = \min\{v(\{i,k\}), w(\{i,k\})\} \). Then \( CSE \) implies that \( \psi_j(v) = \psi_j(w') = \psi_j(w) \) (where \( \psi \) is an \( CSE \) solution).

**References**


