1 Introduction

I study matching problems in which agents are allocated heterogeneous indivisible objects, at most one object to each agent, based on strict priorities for the objects. The priorities can be interpreted as either allocation priorities or ownership priorities. Allocation priorities mean that the agents are supposed to be matched in the order of the priorities to the given object. Ownership priorities mean that the agents have the right to trade the objects in the order of the priorities. There are two well-known matching rules which assign the objects according to these two interpretations, respectively. The first one is the Gale-Shapley deferred acceptance procedure (Gale and Shapley, 1962), and the second one is Gale’s top trading cycle procedure (Shapley and Scarf, 1974), enhanced by “inheritance” (Pápai, 2000). In the following, I will refer to these two important matching rules as the Deferred Acceptance Rule and the Top Trading Cycle Rule.¹ Both of these rules are

¹In this paper, these two rules will always mean the following: the deferred acceptance rule in which agents propose to the objects (as opposed to the objects proposing to the agents), resulting in the agent-
simple and intuitive, and they both provide strong incentives for the agents to reveal their true preferences, as they are both strategyproof in the context of allocating heterogeneous objects.

Given these two well-studied matching rules, and given that both of these rules are strategyproof, my main goal is to explore the possibility of finding other strategyproof rules which combine some of the nice characteristics of the deferred acceptance and top trading cycle rules. It is well-known that the Deferred Acceptance Rule is fair in terms of the priorities, while the Top Trading Cycle Rule is efficient, and that neither rule satisfies the other property. Fairness in this context means that the rule is (core) stable, that is, we cannot find an agent-object pair such that the agent prefers this object to the object she was assigned and, at the same time, has higher priority for the object than the agent who was assigned the object. Note that fairness depends on the priorities for objects. Efficiency, on the other hand, does not depend on the priorities, and requires that the rule prescribe a matching, at every preference profile, which is not Pareto dominated. It is also well known that it is not possible to find a matching rule that is both fair and efficient in addition to strategyproofness (see, for example, Roth and Sotomayor, 1990).

Papers that have studied similar matching problems with priorities include Ergin (2002) and Kesten (2006). These two papers establish that fairness and efficiency can be reconciled only if the priority structure is acyclic, a rather restrictive condition on priorities. They also show that if the priorities are acyclic, then the two matching rules, Deferred Acceptance and Top Trading Cycle, are equivalent, that is, they result in the same matching at every preference profile.

I also start with strict priorities as primitives of the model, but my approach is based optimal matching, and the top trading cycle rule with inheritance, which is a member of the hierarchical exchange rules of Pápai (2000).²

²Fairness is also called no justified envy in the literature.
on a new property of objections to allocations which takes into account the priorities. The objection based on respecting allocation priorities leads to the standard condition of (core) stability or fairness. I introduce another type of objection based on ownership priorities, which considerably weakens the stability requirement. Instead of requiring that an agent $j$ can object if his priority is higher for the object than the agent’s, say agent $i$’s, who is assigned the object, this requirement takes into account the possibility that agent $i$ traded this object with an agent who has a higher priority for the object than $j$ does, and if this possibility cannot be ruled out then $j$ cannot object to allocating the object to $i$, even if $i$ herself has a lower priority for the object than $j$ does. This axiom, which I call No Ownership Objection, can be viewed as a requirement to respect priorities in a minimal sense, and is satisfied by both the Deferred Acceptance Rule and the Top Trading Cycle Rule. I show that No Ownership Objection combined with other axioms, Constrained Efficiency, a mild incentive property called Restricted Individual Monotonicity, and an invariance type of axiom called Weak Nonbossiness, characterize a set of matching rules, what I call the Weak Trading Cycle Rules, that combine properties of both the Deferred Acceptance Rule and the Top Trading Cycle Rule. These rules, however, are not strategyproof. If we strengthen Restricted Individual Monotonicity to Strategyproofness in the previous characterization, then we are back to only the two well-known matching rules. This is my second main result, which strengthens our intuition that strategyproofness cannot be maintained if we combine the two rules in a meaningful manner. Indeed, if we dispense with Weak Nonbossiness, we still end up essentially with only these two matching rules, although in this case a very moderate compromise is possible, but only if the priority table can be partitioned. My results show that we did not overlook any desirable matching rules when studying strategyproofness in priority-based matching. However, it is also significant that if we relax strategyproofness then we can find interesting rules which allow much greater flexibility in choosing the type of priorities than either of the two well-known matching rules does.
2 Matching Problems With Priorities

There is a finite set of players $N = \{1, \ldots, n\}$, with $n \geq 3$, and a finite set of $m \geq 3$ heterogeneous objects $M$. For simplicity of the exposition, I will assume that $n = m$, but the results all extend to the general case where $n \neq m$ is possible. I will call a nonempty subset $S$ of $N$ a coalition. Agents receive at most one object based on priorities for the objects. For each object $a \in M$, there is a strict priority relation denoted by $\Pi_a$, which gives a complete ordered list of the agents. Thus, $i \Pi_a j$ means that $i$ has a higher priority for object $a$ than $j$ does. Let $\Pi = (\Pi_a)_{a \in M}$ denote a set of lists of priorities for the set of objects. I will refer to each set of lists $\Pi$ as a priority table. I will also denote by $\Pi_a(i)$ the rank of agent $i$ for object $a$. Thus, for example, if $i$ is first-ranked for $a$, then $\Pi_a(i) = 1$.

A priority table is partitioned if there exists an ordered list of subsets of $N$, $(S_1, \ldots, S_k)$, where $k \geq 2$, such that $\{S_1, \ldots, S_k\}$ is a partition of $N$, for all $i \in S_1$, $\Pi_a(i) \leq s_1$, and for all $t = 2, \ldots, k$, for all $i \in S_t$, $\sum_{v=1}^{t-1} s_v + 1 \leq \Pi_a(i) \leq \sum_{v=1}^{t} s_v$ for all $a \in M$, where, for all $t = 1, \ldots, k$, $|S_t| = s_t$.

Each agent $i \in N$ has strict preferences $R_i$ over the set of objects, which completely orders the set of objects. Note that each object is preferred to not receiving any objects. I will write $aR_ib$ if either $a$ is strictly preferred to $b$ or $a = b$. $P_i$ denotes strict preferences. Let $R = (R_i)_{i \in N}$ denote a preference profile, and let $\mathcal{R}_i$ and $\mathcal{R}$ be the set of all preferences for agent $i$, and the set of all preference profiles, respectively. I will also use the usual notation of $(R_i, R_{-i})$ for preference profiles. Let $\text{top}(R_i)$ denote a sequence of top-ranked object(s) by $R_i$ in the order of preferences, starting with the highest-ranked object and stopping at any arbitrary object. For example, if $R_i$ ranks $a, b, c, d$ on top, in this order, then (with a slight abuse of notation), $\text{top}(R_i) = (a)$ or $(a, b)$, or $(a, b, c)$, etc. Furthermore, for any $M' \subset M$, let $\text{top}(R_i \setminus M')$ denote the same as $\text{top}(R_i)$, but with respect to the set of objects $M \setminus M'$. 
A (priority-based) matching problem is given by a pair of a priority table and a preference profile, \((\Pi, R)\). Formally, a matching problem is identical to a marriage problem, but the interpretation is different. Only the agents have preferences, while for the objects there are priorities, which will be interpreted in two different ways, as explained in the Introduction. Thus, the priority-based matching problems are not symmetric with respect to the two sides of the market. Throughout this paper, it will be convenient to assume that the priority table is fixed (although arbitrary), so that a priority-based matching problem is simply given by a preference profile \(R\).

Let \(X\) denote the set of all feasible matchings, with generic members \(x, y \in X\). A feasible matching assigns at most one object to each agent, such that the same object cannot be assigned to two different agents. For \(x \in X\), let \(x_i\) denote the object that agent \(i\) receives at matching \(x\). Let \(x_S\) denote the assignment of objects to the members of coalition \(S \subseteq N\) at matching \(x\). Finally, for all \(x \in X\), let \(X_S := \{a \in M : \text{there exists } i \in S \text{ such that } x_i = a\}\).

Given a fixed priority table \(\Pi\), a matching rule is \(f^\Pi : R \to X\). In the following, I will suppress the superscript \(\Pi\) in \(f^\Pi\) for simplicity. Let \(f_i(R)\) denote the object assigned to agent \(i\) at \(R\), and let \(f_S(R)\) denote the assignment of objects to agents in \(S\) at \(R\). Finally, let \(F_S(R) := \{a \in M : \text{there exists } i \in S \text{ such that } f_i(R) = a\}\).

### 3 Allocation and Ownership Objections

I now introduce two types of objections: an allocation objection is based on the allocation interpretation of priorities, while an ownership objection is based on the ownership interpretation of priorities.

There is an allocation objection to \(x\) at \(R\) if there exist agent \(i \in N\) and object \(a \in M\) such that \(aP_i x_i\) and \(iP_a l\), where \(x_l = a\). We will also say that agent \(i\) has an allocation objection at \(R\) (to \(x\) or to \(a\)). If there is no allocation objection at any profile \(R \in \mathcal{R}\) to
A matching rule \( f \), given some matching rule \( f \), then we will say that the matching rule has no allocation objections. It is clear that a matching rule has no allocation objections if and only if it is fair or core stable, and it is satisfied by the Deferred Acceptance Rule, but not the Top Trading Cycle Rule. Thus, it is a very well-known axiom, and we will state it formally as follows. A matching rule \( f \) satisfies **Fairness** if, for all preference profiles \( R \in \mathcal{R} \), there are no \( i, j \in N \) and \( a \in M \) such that \( aP_i f_i(R), i \Pi a j \), and \( f_j(R) = a \).

A **cycle trade** \( y_T \) is an assignment of objects to agents in \( T = \{i_1, \ldots, i_t\}, t \geq 2 \), such that for all \( v = 1, \ldots, t, \) \( i_{v+1} \Pi a_{v+1} i_v \), where \( y_{i_v} = a_{v+1} \). We will say that agent \( i_v \) is the **owner** of \( a_v \), for all \( v = 1, \ldots, t \), in cycle trade \( y_S \). Note that this definition depends on the priorities only and not on the preference profile.

Let \( T \subset N \) such that \( T = \{i_1, \ldots, i_t\} \). A **top-priority path** for \( T \) is a sequence of matched pairs of agents and objects \((j_1a_1, \ldots, j_ka_k)\), where for all \( z = 1, \ldots, k, j_z \in N \) and \( a_z \in M \), such that if we were to remove these matched agents from the market iteratively in the order of this sequence, then the agent who is next in the sequence has top priority for his matched object in the corresponding reduced market. That is, \( j_1 \) has top priority for \( a_1 \), \( j_2 \) has top priority for \( a_2 \) in the market in which \( j_1 \) is removed, etc. Moreover, for all \( v = 1, \ldots, t \), there exists \( b_v \in M \) such that \( i_v \) has top priority for \( b_v \) in the reduced market after \( S = \{j_1, \ldots, j_k\} \) is removed. We will call \( S \) a **top-priority path coalition for** \( T \) (with the corresponding set of objects \( B = \{a_1, \ldots, a_k\} \)) and we will refer to the market reduced by \( S \) as the \( S \)–**reduced market**. Finally, note that if each agent in \( T \) has a top-priority object to begin with, then the \( \emptyset \) is a top-priority path coalition for \( T \).

The second type of objection, ownership objection, strengthens the notion of allocation objection, as follows. There is an **ownership objection** to \( x \) at \( R \) if there exist agent \( i \in N \) and object \( a \in M \) such that

i) \( aP_i x_i \),

ii) \( i \Pi a l \), where \( x_l = a \),

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iii) there are no \( j \) and \( T \subset N \) such that
- \( j \Pi a i \),
- \( j, l \in T \),
- \( x_T \) is a cycle trade for \( T \) in which \( j \) is the owner of \( a \),
- there is a top-priority path for \( T \), with top-priority path coalition \( S \) and corresponding set of objects \( B \), such that \( X_S = B \),
- \( i \notin S \cup T \).

We will also say that agent \( i \) has an ownership objection at \( R \) (to \( x \) or to \( a \)). If there is no ownership objection at any profile \( R \in \mathcal{R} \) to \( f(R) \) for some matching rule \( f \), then we will say that the matching rule has no ownership objections, or, equivalently, that it satisfies the No Ownership Objections axiom. Note that if a matching rule has no allocation objections then it has no ownership objections either, but the converse is not true. Thus, ruling out ownership objections is less demanding than ruling out allocation objections. Indeed, No Ownership Objections is satisfied by both the Deferred Acceptance Rule and the Top Trading Cycle Rule, as this axiom requires that minimal priority rights are respected.

4 Axioms

I present here further axioms besides the priority-based axioms discussed in the last section.

Strategyproofness

A matching rule \( f \) satisfies Strategyproofness if, for all agents \( i \in N \), all preference profiles \( R \in \mathcal{R} \), and all \( R'_i \in \mathcal{R}_i \), we have \( f_i(R)R_i f_i(R'_i, R_{-i}) \).

Restricted Individual Monotonicity

In order to relax the strong requirement of Strategyproofness, I introduce the axiom of Restricted Individual Monotonicity which is implied by Strategyproofness.
For $i \in N$, $R_i \in \mathcal{R}_i$, and $a \in M$, let $\succ_a (R_i)$ denote the strict upper contour set of $a$ at $R_i$: $\succ_a (R_i) := \{c \in M : cP_ia\}$. Given $i$, $R_i$, and $a$, $R_i'$ is a restricted monotonic transformation of $R_i$ for $a$ if there exists $b \in M$ such that $bR_i a$, $\succ_a (R_i') = \succ_b (R_i)$, and for all $c, c' \in \succ_a (R_i')$, $cR_i c'$ if and only if $cR_i c'$.

A matching rule $f$ satisfies Restricted Individual Monotonicity if for all $R \in \mathcal{R}$, $i \in N$, and $R_i' \in \mathcal{R}_i$ such that $R_i'$ is a restricted monotonic transformation of $R_i$ for $f_i(R)$, we have $f_i(R_i', R_{-i}) = f_i(R)$.

Efficiency

A matching rule $f$ satisfies Efficiency if, for all preference profiles $R \in \mathcal{R}$, $f(R)$ is not Pareto dominated. That is, for all $R \in \mathcal{R}$, there is no $x \in X$ such that for all $i \in N$, $x_iR_i f_i(R)$, and for some $j \in N$, $x_jP_j f_j(R)$.

Since Efficiency is not satisfied by the Deferred Acceptance Rule, I will impose a much weaker efficiency axiom that is satisfied by this rule, the well-known property of constrained efficiency.

Constrained Efficiency

A matching rule $f$ satisfies Constrained Efficiency if for all preference profiles $R \in \mathcal{R}$, if there is a Pareto improving trade at $f(R)$ then carrying out this trade would lead to an allocation objection at $R$, which is not present at $f(R)$. More precisely, let $T \subset N$ and let $y_T$ be such that $F_T(R) = \{a \in M : \text{there exists } i \in T \text{ such that } y_i = a\}$ and for all $i \in T$, $y_iP_i f_i(R)$. Then Constrained Efficiency requires that there exist $i \in T$ and $j \in N \setminus T$ such that $j\Pi_i a$ and $aP_j f_j(R)$, where $y_i = a$, and $l\Pi_i j$, where $f_l(R) = a$.

Weak Nonbossiness

The final axiom is an invariance type of axiom, similar to but weaker than nonbossiness, which makes the matching rule more regular and thus our characterizations more tractable.

A matching rule $f$ satisfies Weak Nonbossiness if for all $R$, $i$, $R_i'$, such that $f_i(R) =$
If $T \subset N$ is minimal such that $f_T(R) = f_T(R'_i, R_{-i})$ then either $f_T(R)$ weakly Pareto dominates $f_T(R'_i, R_{-i})$ or $f_T(R'_i, R_{-i})$ weakly Pareto dominates $f_T(R)$, where weak Pareto domination allows for $f_T(R) = f_T(R'_i, R_{-i})$.

5 Priority-Based Matching Rules

Next, we turn to the two main matching rules.

The Deferred Acceptance Rule

Each agent "proposes" to his first-ranked object, and each object that receives multiple proposals "rejects" all but one of the proposing agents based on its priority list. Namely, the agent who has the highest priority among those proposing to the object is the only agent who is not rejected. Each agent who is not rejected by an object is temporarily matched to that object. Each agent who is rejected in this round proposes to his next most favored object on his preference list in the next round, and each object again rejects all but one of the proposing agents (including the agent who is temporarily matched to the object, if any), based on its priority list. Repeat this step until no agent is rejected in a round, and the temporary matches in this round become the final matching.

This rule is the Deferred Acceptance Rule where agents propose to objects (as opposed to the objects proposing to the agents), and the result of this procedure is what is known as the agent-optimal stable matching.

The Top Trading Cycle Rule

Each agent points to the agent who "owns," based on the priorities, the object that they rank first. Identify the top trading cycles, which are simply the cycles of pointing agents (and may consist of one agent pointing to himself). Allocate the objects according to these cycles to the agents in top trading cycles, and remove these agents from the market with
their assigned objects. If any agent who leaves the market at this round owns more than one object, re-assign the ownership of these objects based on the priority table to agents remaining in the market (this is the inheritance of the ownership of objects). Agents who remain in the market keep the objects that they already own. Repeat the same, iteratively, for the remaining market. Note that this rule is well-defined, since there is at least one top trading cycle at each round.

We are going to characterize the following "hybrid" rule, a matching rule which combines characteristics of both of the above rules.

**Weak Trading Cycle Rules**

Each Weak Trading Cycle Rule is associated with a weak order $\tau_a$ of the agents for each object $a$, which is consistent with the priority order $\Pi_a$ (a strict order) of the agents for object $a$. Thus, for each object $a$, $\tau_a$ partitions the set of agents into equivalence classes without contradicting the order in $\Pi$. Thus, if $i \tau_a j$ then it follows that $i \Pi_a j$, but not the other way around.

Given $\tau = (\tau_a)_{a \in M}$, we can describe the corresponding Weak Trading Cycle rule as follows. Each agent points to the agent who "owns," based on the priorities $\Pi = (\Pi_a)_{a \in M}$, the object that he ranks first, identifying the object that he points for. If an object has multiple agents who rank it first then resolve these conflicts, just like in the Deferred Acceptance procedure, but based on $\tau$, not on $\Pi$. That is, if $i$ and $j$ both point because of object $a$ to the same agent, if $i \tau_a j$ then reject agent $j$. However, if $i \Pi_a j$ but $\neg (i \tau_a j)$ then do not reject agent $j$ from object $a$. Continue with this iteratively, just like in the Deferred Acceptance Rule, as long as there are any conflicts that can be resolved based on $\tau$. If no more such conflicts remain, but if there are still conflicts over objects (meaning that more than one agent points to the same object) then identify all top trading cycles, just like in the Top Trading Cycle Rule. Allocate the objects according to these cycles to the agents.
in top trading cycles, and remove these agents from the market with their assigned objects. If any agent who leaves the market at this round owns more than one object, re-assign the ownership of these objects based on the priority table II to agents remaining in the market. This inheritance of the ownership of objects is the same as in the Top Trading Cycle Rule. Agents who remain in the market keep the objects that they already own. Now repeat the same, iteratively, for the remaining market. That is, resolve all conflicts first based on \( \tau \), and if any conflicts remain then make final assignments based on top trading cycles, and so on. Note that this rule is well-defined, since there is at least one top trading cycle at each round.

In the following, we will refer to rounds of this iterative procedure as either a DA (Deferred Acceptance) round or a TTC (Top Trading Cycle) round. Also, given an arbitrary set of preference relations (i.e., not necessarily a weak order) \( \hat{\tau} = (\hat{\tau}_a)_{a \in M} \) that is consistent with \( (\Pi_a)_{a \in M} \), we will refer to the well-defined matching rule that proceeds exactly as a Weak Trading Cycle Rule based on \( \hat{\tau} \) as a Partial Trading Cycle Rule.

### 6 Matching Rules With Minimal Priority Rights

We are now ready to present the results. The first theorem characterizes the set of Weak Trading Cycle Rules.

**Theorem 1** A matching rule satisfies No Ownership Objection, Restricted Individual Monotonicity, Constrained Efficiency, and Weak Nonbossiness if and only if it is a Weak Trading Cycle Rule.

It turns out that if we require strategyproofness instead of the much weaker Restricted Individual Monotonicity, then we are back to the two main matching rules and no compromise is possible. This result also confirms that we had not overlooked any interesting strategyproof rules in the past which are based on priorities.
Theorem 2 A matching rule satisfies No Ownership Objection, Strategyproofness, Constrained Efficiency, and Weak Nonbossiness if and only if it is either the Deferred Acceptance Rule or the Top Trading Cycle Rule.

In the next theorem we dispense with the axiom of Weak Nonbossiness while keeping Strategyproofness. In this characterization we end up essentially with the two main matching rules, although a very moderate compromise is possible in this case, as described below. A matching rule is called a **Partitioned Top Trading Cycle-Deferred Acceptance Rule** if either the Top Trading Cycle Rule or the Deferred Acceptance Rule is applied to each partition of \( \Pi \), sequentially. More precisely, let the ordered list of subsets of \( N \) corresponding to the partition of \( \Pi \) be \((S_1, \ldots, S_k)\). Then the rule applies either the Top Trading Cycle Rule or the Deferred Acceptance Rule to agents in \( S_1 \), after which the matched agents and objects are removed, and then again either the Top Trading Cycle Rule or Deferred Acceptance Rule is applied to \( S_2 \), and so on, for each coalition \( S_t, t = 1, \ldots, k \), in the partition of \( \Pi \). Moreover, for each matching rule the order in which the Top Trading Cycle Rule or the Deferred Acceptance Rule is chosen is not fixed a priori, except for the first coalition in the partition of \( \Pi \), as the choice of the Deferred Acceptance Rule or Top Trading Cycle Rule for subsequent coalitions may depend on the assignments of the agents in the previous coalitions in the partition.

Note that if \( s_t \in \{1, 2\} \) for some \( t \in \{1, \ldots, k\} \) then the two matching rules are equivalent for the coalition \( S_t \). Thus, if \( s_t \in \{1, 2\} \) for all \( t = 1, \ldots, k \) then the two matching rules are equivalent. Furthermore, if \( s_t = 1 \) for all \( t = 1, \ldots, k \) then the matching rule is a serial dictatorship. Finally, let me note that if the priority table is not partitioned then the only matching rules that satisfy the two axioms in the theorem are the Deferred Acceptance Rule and the Top Trading Cycle Rule.

Our last characterization theorem is the following.
Theorem 3 A matching rule satisfies No Ownership Objection, Strategyproofness, and Constrained Efficiency, if and only if it is a Partitioned Top Trading Cycle-Deferred Acceptance Rule.

7 Proofs

Let \( \text{top}(R_i, a) \) denote the sequence of objects ranked on top by \( R_i \), ending with object \( a \). For example, if \( R_i \) ranks objects in the order of \( d \) first, then \( b, e, a, c \), then \( \text{top}(R_i, a) = (d, b, e, a) \). Fix \( i \in N, R_i \in \mathcal{R}_i \), and \( a \in M \). Then \( R'_i \) is an identical monotonic transformation of \( R_i \) for \( a \) if \( \text{top}(R'_i, a) = \text{top}(R_i, a) \).

Lemma 1 If a matching rule \( f \) satisfies Restricted Individual Monotonicity, Constrained Efficiency, and Weak Nonbossiness then for all \( R, i, \) and \( R'_i \) such that \( R'_i \) is an identical monotonic transformation of \( R_i \) for \( f_i(R) \), we have \( f(R'_i, R_{-i}) = f(R) \).

Proof: Since in identical monotonic transformation is a restricted monotonic transformation, it follows from Restricted Individual Monotonicity that \( f_i(R'_i, R_{-i}) = f_i(R) \). Then \( F_{N\setminus\{i\}}(R) = F_{N\setminus\{i\}}(R'_i, R_{-i}) \), and given that \( R'_i \) is an identical monotonic transformation of \( R_i \), Weak Nonbossiness implies that Constrained Efficiency is violated, unless \( f(R'_i, R_{-i}) = f(R) \).

Proof of Theorem 1:

A bottom transformation of \( R_i \) for \( M' \subset M \) is \( R'_i \) that preserves all the preference orderings of \( R_i \), except that it ranks objects in \( M' \) last.

Step 1 Identification of \( \tau(R) \) for all \( R \in \mathcal{R} \).
A pivotal trading coalition $T$ for $(j, l, a)$, given $j \Pi a l$, has the following properties:

i) $l \in T$,

ii) $j \not\in T$, and

iii) there exists a cycle trade $x_T$ for $T$ in which $x_l = a$, and

iv) $w \Pi a j$, where agent $w \in N$ owns $a$ in $x_T$.

Specifically, let $T = \{1, \ldots , t\}$, $t \geq 2$, $l = 1$, $w = 2$, and for all $v \in T$, $x_v := a_v$. Also, for all $v \in T$, mod $t$, $v$ owns $a_{v-1}$ in $x_T$.

A pivotal profile $\tilde{R}$ for $(R, j, l, a)$ satisfies the following with respect to some pivotal trading coalition $T$ for $(j, l, a)$ and to some top-priority coalition $S$ for $T$, with $T \cap S = \emptyset$, such that for all $v \in T$, mod $t$, $v$ has top priority for $a_{v-1}$ in the $S$-reduced market. Let the top-priority path for $T$ corresponding to $S$ be $(l_1 b_1, \ldots , l_k b_k)$, let $S = \{l_1, \ldots , l_k\}$, and let $B = \{b_1, \ldots , b_k\}$. Let $F_S(R) = B$ such that for all $z = 1, \ldots , k$, $f_{l_z}(R) = c_z$.

i) for all $z = 1, \ldots , k$, top$(\tilde{R}_{l_z}) = (c_z, b_z)$,

ii) for all $v \in T$, top$(\tilde{R}_v \setminus B) = (a_v, a_{v-1})$,

iii) top$(\tilde{R}_j \setminus B) = a$,

iv) for all $v \in N \setminus (T \cup S \cup \{j\})$, $\tilde{R}_v$ is a bottom transformation of $R_v$ for $\{a_1, \ldots , a_t\}$.

Let $j \tau_R a l$ if one of the following is satisfied:

1) there is no pivotal profile for $(R, j, l, a)$.

2) for all pivotal profiles $\tilde{R}$ for $(R, j, l, a)$, $f_l(\tilde{R}) \neq a$.

Let $\neg (j \tau_R a l)$ if there exists a pivotal profile $\tilde{R}$ for $(R, j, l, a)$ such that $f_l(\tilde{R}) = a$.

**Step 2** For all $R, \tilde{R}$, and for all $j, l$, and $a$, $j \tau_R a l$ if and only if $j \tau_{\tilde{R}} a l$.

Fix $i \in N$, $R \in \mathcal{R}$, and $R'_i \in \mathcal{R}_i$ such that $R_i \neq R'_i$. Fix $j, l \in N$, $j \neq l$ and $a \in M$. We will show that $j \tau_R a l$ if and only if $j \tau_{R'_i} a l$, where $R' := (R'_i, R_{-i})$.

Suppose that $j \tau_R a l$ and $\neg (j \tau_{R'} a l)$. Since $\neg (j \tau_{R'} a l)$, there exists a pivotal profile $\tilde{R}'$ for $(R', j, l, a)$ such that $f_l(\tilde{R}') = a$. Let $T$ be the pivotal trading coalition for $(j, l, a)$ such
that $\tilde{R}'$ is a pivotal profile for $(R', j, l, a)$ with respect to $T$. Let $T$ satisfy all the properties described in the definition of a pivotal trading coalition. Let $S$ be a top-priority coalition for $T$ such that $\tilde{R}'$ is a pivotal profile for $(R', j, l, a)$ with respect to $S$. Let the corresponding top-priority path be $(l_1b_1, \ldots, l_kb_k)$, so that $S = \{l_1, \ldots, l_k\}$. Note that $S \cap T = \emptyset$. Let $\tilde{R}$ be a pivotal profile for $(R, j, l, a)$ with respect to $T$ and $S$ such that $\tilde{R}_j = \tilde{R}'_j$, for all $v \in T \cup S$, $\tilde{R}_v = \tilde{R}'_v$, and for all $v \in N \setminus \{i\}$, $\tilde{R}_v = \tilde{R}'_v$. This implies that $i \not\in (T \cup S)$ and $i \neq j$, given that $R_i \neq R'_i$.

Note that $F_S(\tilde{R}) = B$ and $F_S(\tilde{R}') = B$. Then, since for all $v = 1, \ldots, t$, mod $t$, top($\tilde{R}_v \setminus B$) = $(a_v, a_{v-1})$, and $v$ has top priority for $a_{v-1}$ in the $S$-reduced market, No Ownership Objection implies that for all $v = 1, \ldots, t$, mod $t$, $f_v(\tilde{R}) \in \{a_v, a_{v-1}\}$. Similarly, for all $v = 1, \ldots, t$, mod $t$, $f_v(\tilde{R}') \in \{a_v, a_{v-1}\}$. Since $f_i(\tilde{R}) = a$, for all $v \in T$, $f_v(\tilde{R}) = a$. Moreover, since $f_i(\tilde{R}) \neq a$, which follows from $-(j^\tau_a R | l)$, for all $v \in T$, mod $t$, $f_v(\tilde{R}) = a_{v-1}$. Given that $i \not\in (T \cup S \cup \{j\})$, $\tilde{R}_i$ is a bottom transformation of $R_i$ for $\{a_1, \ldots, a_t\}$, and $\tilde{R}'_i$ is a bottom transformation of $R'_i$ for $\{a_1, \ldots, a_t\}$. Note also that $F_T(\tilde{R}) = F_T(\tilde{R}') = \{a_1, \ldots, a_t\}$, and $f_T(\tilde{R}')$ Pareto dominates $f_T(\tilde{R})$. Since for all $v \in T$, $f_i(\tilde{R})\hat{R}_i a_v$, Constrained Efficiency is violated. Therefore, for all $i, R, R'_i, j, l, a$ we have $j^\tau_a R | l$ if and only if $j^a_{\tau_1}(R'_i, R, l)$. Finally, note that since $i, R, R'_i, j, l, a$ were chosen arbitrarily, this implies that for all $R, \tilde{R}$, and for all $j, l, a$, $j^\tau_a R | l$ if and only if $j^a_{\tau_1} l$. Therefore, for all $R, \tilde{R}$, $\tau R = T \tilde{R}$.

**Lemma 2** Let a matching rule $f$ satisfy No Ownership Objection, Constrained Efficiency, Restricted Individual Monotonicity, and Weak Nonbossiness. Let $T \subset N$ be a top trading cycle coalition in round $h$ of the Partial Trading Cycle procedure based on $\tau$ applied to a preference profile $R \in \mathcal{R}$ such that members of $T$ are assigned their corresponding traded object in round $h$. Assume that $f(R)$ does not contain this top trading by $T$ in round $h$. Then there exist $j \in N \setminus T$ and $l \in T$, where $l$ gets $a \in M$ in the top trading of $T$ in round $h$ and $f_l(R) \neq a$ such that $j$ and $l$ both point to $a$ in round $h$ of the Partial Trading Cycle procedure applied to $R$ and $j^\tau a l$. 

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**Step 3** Let $y \in X$ be the result of the Partial Trading Cycle procedure at some $R \in \mathcal{R}$, based on $\tau$. Then $f(R) = y$.

Determine $\tau$ as defined in Step 1, which can be done using any preference profile, given Step 2. Note that for all $a \in M$, $\tau_a$ is an arbitrary preference relation on $N$ that is consistent with $\Pi_a$. Fix $R \in \mathcal{R}$. Let $y \in X$ be the result of the Partial Trading Cycle procedure at $R$, based on $\tau$. Suppose $f(R) \neq y$. We will prove that this leads to a contradiction.

Let $h$ be the first round in which one can observe that $f(R) \neq y$. That is, in any TTC round among rounds 1 to $h - 1$, each agent $i$ who is removed from the market with a matched object satisfies $f_i(R) = y_i$. Furthermore, in any round among rounds 1 to $h - 1$, if agent $i$ is rejected from object $a$ then $aP_i f_i(R)$. Let $l \in N$ such that $f_l(R) \neq y_l$ and this can be seen in round $h$ of the procedure for the first time.

**Case 1**: Agent $l$ gets rejected from $f_l(R)$ in round $h$. Note that this means that $f_l(R)P_l y_l$.

**Case 1a**: Round $h$ is a DA round.

Let $f_l(R) = a$. Then there exists $j \in N$ such that $j\Pi_a l$, $j\tau_a l$, and $j$ and $l$ both point to object $a$ in round $h$.

Suppose that $f_j(R)P_j a$. Given that $j$ points to $a$ in round $h$, this implies that $j$ got rejected from $f_j(R)$ in round $h' < h$, contrary to our assumption. Thus, $aR_j f_j(R)$. Since $f_l(R) = a$, this means that $aP_j f_j(R)$. Since $j\Pi_a l$ and $aP_j f_j(R)$, No Ownership Objection implies that there exists $T \subset N$ and $w \in T$ such that

- $w\Pi_a j$,
- $l \in T$,
- $f_T(R)$ is a cycle trade for $T$ in which $w$ is the owner of $a$,
- there is a top-priority path for $T$, with top-priority path coalition $S$ and corresponding set of objects $B$, such that $F_S(R) = B$,
- $j \notin S \cup T$.  

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Let the relevant top-priority path be given by \((l_1 b_1, \ldots, l_k b_k)\), so that \(S = \{l_1, \ldots, l_k\}\) and \(B = \{b_1, \ldots, b_k\}\). Note that \(S \cap T = \emptyset\). Let \(T = \{1, \ldots, t\}\). For all \(v \in T\), mod \(t\), let \(f_v(R) = a_v\), and let the object owned by \(v\) in \(f_T(R)\) be \(a_{v-1}\). Then, letting \(l = 1\), we have \(a = a_1\) and \(w = 2\).

Now let’s determine a modified set of preference relations \((\tilde{\tau}_a)_{a \in M}\) based on \(\tau\), which is also consistent with \(\Pi\). This will determine, in turn, a modified Partial Trading Cycle procedure at \(R\), based on \(\tilde{\tau}\). First, if \(T\) is a top trading cycle coalition in round \(h\) of the procedure at \(R\) based on \(\tau\) then \(\tilde{\tau} = \tau\). Otherwise, let \(\tilde{\tau}\) be such that all preference relations are negated in \(\tau\) regarding the cycle trade \(f_T(R)\), starting with \(\neg(j \tilde{\tau}_a l)\), so that based on this suitably modified \(\tilde{\tau}\), there exists \(\tilde{h} \geq h\) such that in this modified procedure at \(R\), based on \(\tilde{\tau}\), \(f_T(R)\) is a top trade for \(T\) in round \(\tilde{h}\). Note that then all top trades in \(S\) take place in this procedure no later than round \(\tilde{h} - 1\). Note also that there may be top trading cycle coalitions in rounds 1 to \(\tilde{h} - 1\) other than coalitions in \(S\), and the corresponding top trades are all included in \(f(R)\).

Let \(S' \subset N\) be the set of agents who are matched in TTC rounds before round \(\tilde{h}\) in the modified procedure at \(R\) based on \(\tilde{\tau}\), excluding agents in \(S\). Thus, \(S \cap S' = \emptyset\). Note also that \(S' = \emptyset\) is possible. Let us also remark that \(j \notin S'\) since \(j\) is still in the market in round \(\tilde{h}\). Let \(\hat{S} = S \cup S'\).

Let \(R'\) satisfy the following:

i) For all \(v \in T\), let \(R'_v\) be the restricted monotonic transformation of \(R_v\) for \(a_{v-1}\) such that \(R'_v\) ranks \(a_{v-1}\) directly after \(a_v\).

ii) For all \(z = 1, \ldots, k\), let \(R'_l_z\) be the restricted monotonic transformation of \(R_{l_z}\) for \(b_z\) such that \(R'_l_z\) ranks \(b_z\) directly after \(c_z\) if \(b_z \neq c_z\), and otherwise let \(R'_l_z := R_{l_z}\).

iii) For all \(i \in S'\), let \(R'_i\) be the restricted monotonic transformation of \(R_i\) for \(e_i\), where \(e_i\) is the object traded by \(i\) in the modified procedure at \(R\). More specifically, let \(R'_i\) rank \(e_i\) directly after \(f_i(R)\) if \(e_i \neq f_i(R)\), and otherwise let \(R'_i := R_i\).
iv) For all $i \in N \setminus (T \cup S')$, let $R'_i := R_i$.

Note that for all $i \in N$, $R'_i$ is an identical monotonic transformation of $R_i$ for $f_i(R)$, and thus Lemma 1 implies that $f(R') = f(R)$.

Let $B'$ denote the set of objects traded by $S'$ in the modified procedure. Note that $B \cap B' = \emptyset$. Let $\hat{B} = B \cup B'$.

Below we use the following notation for simplicity: $\text{top}(R_i) = (a, b)$ means $\text{top}(R_i) = a = b$ if $a = b$.

Let $\bar{R}$ satisfy the following:

i) For all $v = 1, \ldots, t$, let $\text{top}(\bar{R}_v) = (a_v, a_{v-1})$.

ii) For all $z = 1, \ldots, k$, let $\text{top}(\bar{R}_l) = (c_z, b_z)$.

iii) For all $i \in S'$, let $\text{top}(\bar{R}_i) = (f_i(R), e_i)$.

iv) For all $i \in N \setminus (T \cup \hat{S})$, let $\bar{R}_i = R'_i$.

We will show that $f(T \cup \hat{S}'(\bar{R})) = f(T \cup \hat{S}'(R'))$.

Observation 1: Let $i \in T \cup \hat{S}$. If $\bar{R}_i$ is such that for all $b \in B$, the upper contour set of $b$ is not larger at $\bar{R}_i$ than at $R_i$ and if $\succ_{f_i(R)}(\bar{R}_i) = \succ_{f_i(R)}(R'_i)$ then $f_i(R) = f_i(R'_i, R-i)$.

Let $D \subset M$ denote the following set of objects:

$$D := \left\{ d \in M : \text{ there exists } i \in T \cup \hat{S} \text{ such that } dP'_i(f_i(R')) \right\}.$$ 

Let $\hat{D} = D \setminus B$. Note that if $D = \emptyset$ then we have $f(T \cup \hat{S}'(\bar{R})) = f(T \cup \hat{S}'(R'))$. Indeed, we also have $f(\bar{R}) = f(R')$ in this case. Now fix $d \in \hat{D}$ and let $i$ have the lowest priority for $d$ among all agents in $\hat{S}$ who rank $d$ above $f_i(R')$. Let $\hat{R}_i$ satisfy $\succ_{f_i(R')} (\hat{R}_i) = \succ_{f_i(R')} (\bar{R}_i)$ and let $d$ be ranked directly above $f_i(R')$, while all other preference orderings are preserved compared to $R'_i$. Then Observation 1 implies that $f_i(R') = f_i(\hat{R}_i, R-i')$. Now let $\hat{R}_i$ be the same as $\bar{R}_i$, except that the order of $d$ and $f_i(R')$ is reversed, so that $\hat{R}_i$ ranks $f_i(R')$ directly above $d$. Then $\hat{R}_i$ is a restricted monotonic transformation of $\bar{R}_i$ for $f_i(R')$, and Restricted Individual Monotonicity implies that $f_i(\hat{R}_i, R-i) = f_i(\bar{R}_i, R-i)$. 

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Hence, $f_i(\hat{R}_i, R_{-i}) = f_i(R')$. This means that $F_{N\setminus\{i\}}(\hat{R}_i, R_{-i}) = F_{N\setminus\{i\}}(R'_i)$ and Weak Nonbossiness and Constrained Efficiency imply that either $f_{N\setminus\{i\}}(\hat{R}_i, R_{-i}) = f_{N\setminus\{i\}}(R'_i)$ or there exists $d' \in F_{N\setminus\{i\}}(R')$ such that, given $f_i(\hat{R}_i, R_{-i}) = d'$ we have $i\Pi d'$ and $d' P_i' f_i(R')$. If $d' \neq d$ then $d' P_i f_i(\hat{R}_i, R_{-i})$ and $i$ has an allocation objection to $d'$ at $f_i(\hat{R}_i, R_{-i})$. But this is a contradiction since $d' \notin B$. Therefore, $d' = d$. But then, since $i$ has the lowest priority for $d$ among all agents in $\hat{S}$ who rank $d$ above $f_i(R')$, $i \notin \hat{S}$. Thus, $f_{T \cup \hat{S}}(\hat{R}_i, R_{-i}) = f_{T \cup \hat{S}}(R')$.

We can continue iteratively in the same manner until the only objects ranked above $f_i(R')$ for any $i \in T \cup \hat{S}$ are objects in $B$. Then, by an iterative application of the No Ownership Objection axiom, we find that $f_{T \cup \hat{S}}(\hat{R}) = f_{\hat{T} \cup \hat{S}}(R')$.

Now let $\hat{R}$ satisfy the following.

i) For all $i \in T \cup \hat{S}$, let $\hat{R}_i := \hat{R}_i$.

ii) Let $\hat{R}_j := \hat{R}_j$.

iii) For all $i \in N \setminus (T \cup \hat{S} \cup \{j\})$, let $\text{bottom}(\hat{R}_i) = (a_1, \ldots, a_t)$.

Then No Ownership Objection implies that $f_{T \cup \hat{S}}(\hat{R}) = f_{\hat{T} \cup \hat{S}}(\hat{R})$. Now note that $\hat{R}$ is a pivotal profile for $(\hat{R}, j, l, a)$ for some preference profile $\hat{R} \in R$ with respect to the pivotal trading coalition $T$ for $(j, l, a)$ and with respect to the top-priority coalition $\hat{S}$ for $T$, where $\hat{S} \cap T = \emptyset$. Note, in particular, that since $j$ points to $a$ in round $h$ of the procedure, $\text{top}(R_j \setminus B) = \text{top}(\hat{R}_j \setminus B) = a$. Since $f_i(\hat{R}) = a$ it follows that $\neg(j \tau_a^g l)$. By Step 2, this means that $\neg(j \tau_a l)$, which is a contradiction.

**Case 1b:** Round $h$ is a TTC round.

Then there exists $T \subset N$ that is a top trading cycle coalition in round $h$ of the partial trading cycle procedure at $R$ such that $l \notin T$, $i \in T$, and $i$ gets $a$ in the top trading corresponding to $T$. Moreover, since both $i$ and $l$ point to $a$ in round $h$, we can conclude that $\neg(l \tau_a^R i)$. Similarly, no agent $v$ exists who points to $b$ in round $h$ such that $v \tau_b^R v'$, where $v' \in T$ and $v'$ gets object $b$ in the top trade of $T$ in round $h$. This, however, contradicts Lemma 2.
Case 2: For all \( i \in N \), \( y_i R_{i} f_i(R) \).

In this case there exists a top trading cycle coalition \( T \subset N \) in some round, say \( h \), of the partial trading cycle procedure at \( R \) such that for all \( i \in T \), \( y_i P_{i} f_i(R) \) and \( y_T \) is a simple reallocation of the objects in \( F_T(R) \). Then, since \( f(R) \) does not include this top trade, Lemma 2 implies that there exist \( j \in N \setminus T \) and \( l \in T \) such that \( y_l = a \), \( j \Pi_a l \), \( j \) and \( l \) both point to \( a \) in round \( h \), and \( j \tau_a^{RL} l \). This, however, contradicts \( y_l = a \).

We checked each possible case and arrived at a contradiction in each case. Thus, \( f(R) = y \), as desired.

Step 4

For all \( a \in M \), \( \tau_a \) satisfies transitivity and negative transitivity.

Transitivity:
Let \( j \tau_a l \) and \( l \tau_a i \) for some \( j, l, i \in N \) and \( a \in M \). Suppose that \( \neg (j \tau_a i) \). Note that \( j \Pi_a l \Pi_a i \).

Moreover, since \( \neg (j \tau_a i) \), \( j \) does not have first priority for \( a \), by No Ownership Objection.

Let \( w \in N \) have first priority for \( a \). Without loss of generality, let \( b, c, d \in M \) such that \( \Pi_b(i) = 1 \), \( \Pi_c(l) = 1 \), and \( \Pi_d(j) = 1 \). Let \( R \) satisfy the following:
\begin{enumerate}
  \item \( \text{top } (R_w) = (b, c, d) \),
  \item \( \text{top } (R_i) = (a, b) \),
  \item \( \text{top } (R_l) = (a, c) \),
  \item \( \text{top } (R_j) = (a, d) \).
\end{enumerate}

Then, since \( j \tau_a l \) and \( l \tau_a i \), No Ownership Objection implies that \((w, i, l, j)\) is assigned \((d, b, c, a)\) at \( f(R) \).

Now let \( R' \) satisfy the following:
\begin{enumerate}
  \item \( \text{top } (R_l) = (c) \),
  \item \( R_{N \setminus \{l\}} = R'_{N \setminus \{l\}} \).
\end{enumerate}

Then, since \( \neg (j \tau_a i) \) and \( l \tau_a i \), \((w, i, l, j)\) is assigned \((b, a, c, d)\) at \( f(R) \). Given that \( f_l(R') = f_l(R) \), this contradicts Weak Nonbossiness.
Negative transitivity

Let $\neg(j \tau a i)$ and $\neg(l \tau a i)$ for some $j, l, i \in N$ and $a \in M$. Suppose that $j \Pi_a i$. Note that $j \Pi_a l$. Moreover, since $\neg(j \tau a l)$, $j$ does not have first priority for $a$, by No Ownership Objection. Let $w \in N$ have first priority for $a$. Without loss of generality, let $b, c, d \in M$ such that $\Pi_b(i) = 1$, $\Pi_c(l) = 1$, and $\Pi_d(j) = 1$. Let $R$ satisfy the following:

i) $\text{top}(R_w) = (b, c)$,

ii) $\text{top}(R_i) = (a, b)$,

iii) $\text{top}(R_l) = (a, c)$,

iv) $\text{top}(R_j) = (a, d)$.

Then, since $j \tau a i$ and $\neg(j \tau a l)$, No Ownership Objection implies that $(w, i, l, j)$ is assigned $(c, b, a, d)$ at $f(R)$.

Now let $R'$ satisfy the following:

i) $\text{top}(R'_d) = (d)$,

ii) $R_{N \setminus \{d\}} = R'_{N \setminus \{d\}}$.

Then, since $\neg(l \tau a i)$, $(w, i, l, j)$ is assigned $(b, a, c, d)$ at $f(R)$. Given that $f_d(R') = f_d(R)$, this contradicts Weak Nonbossiness.

Proof of Theorem 2:

Let $j, l, i \in N$ and $a \in M$ such that $j \Pi_a l \Pi_a i$, $\neg(j \tau a l)$, and $l \tau a i$. We will show that the Weak Trading Cycle Rule associated with $\tau$ does not satisfy Strategyproofness. Since $\neg(j \tau a l)$, $j$ does not have first priority for $a$, by No Ownership Objection. Let $w \in N$ have first priority for $a$. Let $b \in M$ such that $j \Pi_b w$ and $j \tau_b w$. Then, if $\text{top}(R_j) = (b, a)$ then we can specify $R$ such that there is a top trading cycle in some round of the Weak Trading Cycle Rule at $R$ in which $w$ is assigned $b$. Then $f_w(R) = a$ and $a P_j f_j(R)$. Now if $\text{top}(R'_j) = a$ then $f_j(R'_j, R_{-j}) = a$, and strategyproofness is violated. This implies that either for all $a \in M$ and all $j, l$ such that $\Pi_a(j) \neq 1$ and $j \Pi_a l$, we have $j \tau a l$ or for all $a \in M$
and all \( j, l \) such that \( \Pi_a(j) \neq 1 \) and \( j\Pi_a l \), we have \( \neg(j\tau_al) \). In the former case \( f \) is the Deferred Acceptance Rule, while in the latter case \( f \) is the Top Trading Cycle Rule.

**Proof of Theorem 3:**

**Lemma 3** If the priority table \( \Pi \) cannot be partitioned then No Ownership Objection, Strategyproofness, and Constrained Efficiency imply Weak Nonbossiness.

Given Lemma 3, we can apply Theorem 2 iteratively to each partition of \( \Pi \) in order to prove that \( f \) is a Partitioned Top Trading Cycle-Deferred Acceptance Rule.

**References**


