Every hierarchy of beliefs is type*

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Abstract

Any model of incomplete information situations has to consider the players' hierarchies of beliefs, which can make the modeling very cumbersome. Harsányi [12] suggested that the hierarchies of beliefs can be replaced by types, i.e., a type space can substitute for the hierarchies of beliefs (henceforth Harsányi program). In the purely measurable framework Heifetz and Samet [15] formalized the concept of type space, and proved that there is universal type space, i.e., the most general type space exists. Later Meier [17] showed that the universal type space is complete, in other words, the universal type space is a consistent object. After these results, only one step is missing to prove that the Harsányi program works, that every hierarchy of beliefs is in the complete universal type space, put it differently, every hierarchy of beliefs can be replaced by type. In this paper we also work in the purely measurable framework, and show that the types can substitute for all hierarchies of beliefs, i.e., the Harsányi program works.

1 Introduction

It is recommended that the models of incomplete information situations be able to consider the players' hierarchies of beliefs, e.g. player 1's beliefs about the parameters of the game, player 1's beliefs about player 2's beliefs about the parameters of the game, player 1's beliefs about player 2's beliefs about player 1's beliefs about the parameters of the game, and so on. However the explicit use of hierarchies of beliefs[1] makes the analysis very cumbersome, hence it is desirable that they do not appear explicitly in the models.

In order to make the models of incomplete information situations more handy, Harsányi [12] suggested that the hierarchies of beliefs could be replaced by types. He wrote[2] "It seems to me that the basic reason why the theory of games with incomplete information has made so little progress so far lies in the fact that these games give rise, or at least appear to rise, to infinite regress in

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[1]In this paper we use the terminology hierarchy of beliefs instead of the longer coherent hierarchy of beliefs.

[2][12] pp.163–167.

reciprocal expectations on the part of the players. ... The purpose of this paper is to suggest an alternative approach to the analysis of games with incomplete information. ... As we have seen, if we use the Bayesian approach, then the sequential-expectations model for any given I-game G will have to be analyzed in terms of infinite sequences of higher and higher-order subjective probability distributions, i.e., subjective probability distributions over subjective probability distributions. In contrast, under own model, it will be possible to analyze any given I-game G in terms of one *unique* probability distribution R^* (as well as certain conditional probability distributions derived from R^*). ... Instead of assuming that certain important *attributes* of the players are determined by some hypothetical random events at the beginning of the game, we may rather assume that the players *themselves* are drawn at random from a certain hypothetical population containing the mixture of different "types", characterized by different attribute vectors (i.e., by different combinations of relevant attributes). ... Our analysis of I-games will be based on the assumption that, in dealing with incomplete information, every player i will use Bayesian approach. That is, he will assign a *subjective* joint probability P_i to all variables unknown to him − or at least to all unknown *independent* variables, i.e., to all variables no depending on the players' own strategy choices."

In other words, Harsányi's main concept was that the types can substitute for the hierarchies of beliefs, and all types can be collected into an object on which the probability measures are for the players' (subjective) beliefs. Henceforth, we call this method of modeling Harsányi program.

However, at least two questions come up in connection with the Harsányi program: (1) is the concept of type itself appropriate for the proposal under consideration? (2) can every hierarchy of beliefs be a type?

Question (1) consists of two subquestions. First, can all types be collected into one object? The concept of universal type space formalizes this requirement: the universal type space in a certain category of type spaces is a type space (a) which is in the given category, and (b) into which, every type space of the given category can be mapped in a unique way. In other words, the universal type space is the most general type space, it contains all type spaces (all types). In the purely measurable framework Heifetz and Samet [15] introduced the idea of (universal) type space, and proved that the universal type space exists.

Second, can every probability measure on the object of the collected types (type space) be a (subjective) belief? Brandenburger [5] introduced the notion of complete type space: a type space is complete, if the type functions in it are surjective (onto). Put it differently, a type space is complete, if all probability measures on the object consisting of the types of the model are assigned to types. Quite recently Meier [17] showed that the purely measurable universal type space is complete. Summing up the above discussion, the answer for question (1) is affirmative, i.e., in the purely measurable framework the complete universal type space exists.

Question (2) is on that whether or not the universal type space contains every hierarchy of beliefs. Mathematically the problem is the following: every hierarchy of beliefs defines an inverse system of measure spaces, and the question is that: do these inverse systems of measure spaces have inverse limits? Kolmogorov Extension Theorem is on this problem, however it calls for topological concepts, e.g. for inner compact regular probability measures. Therefore up to now, all papers on this problem (e.g. Mertens and Zamir [20], Brandenburger and Dekel [7], Heifetz [13], Mertens et al. [21], Pintér [23] among others) used topological type spaces instead of purely measurable ones. Although these papers give positive answer for question (2) (i.e. their type spaces contain all "considered" hierarchies of beliefs), very recently Pintér [24] showed that there is no universal topological type space (there is no such a topological type space that contains every topological type space), therefore the answer for question (1) is negative in this case i.e., in the topological framework the Harsányi program breaks down.

In the above mentioned papers the authors answer question (2) (affirmatively) by constructing an object consisting all considered hierarchies of beliefs, called beliefs space, and show that the constructed beliefs space defines (is equivalent to) a topological type space.

In this paper we work with the category of type spaces introduced by Heifetz and Samet, i.e., in the purely measurable framework. It is our main result that (in the purely measurable framework) every hierarchy of beliefs is type, put it differently, the Harsányi program works. The strategy of the proof is the same as in the above papers, i.e., we construct such an object that contains every hierarchy of beliefs (see Definition 13) and generates a type space. More exactly, it is showed that the (purely measurable) beliefs space is equivalent to the complete universal type space.

As we have already mentioned the above strategy depends on the Kolmogorov Extension Theorem. Since we work in the purely measurable framework, therefore we avoid the use of topological concepts and use a non-topological variant of the Kolmogorov Extension Theorem. Mathematically, we use a new result of Pintér [25] to show that the inverse systems of measure spaces under consideration have inverse limits.

One important remark, our result does not contradict with Heifetz and Samet's [16] counterexample, since their non-type hierarchy of beliefs is not in the purely measurable beliefs space (for the the details see Section 6).

The paper is organized as follows: in the first section we introduce an example illustrating our main result. Section 5. presents the technical setup and some basic results of the field. Our main result (Theorem 14) comes up in Section 4. Section 5. is on the proof of Theorem 14. Section 6. is for a detailed discussion of the connection between our result and two other papers Heifetz and Samet [16], and Pintér [24]. The last section briefly concludes.

2 An example

In this section we introduce an example for illustrating the importance of the hierarchies of beliefs.

Consider a 2×2 game in strategic form, two players: Player 1 and Player 2, both have two actions U, D and L, R respectively, there are two states of the nature in the model: s_1 and s_2 ($S = \{s_1, s_2\}$) with the payoffs in Tables 1 and 2.

In this example by an action is rationalizable for a certain player we mean that there is such a state of the world that the common belief of rationality implies that the player under consideration plays the given action. Although

		Player 2	
		L	R
Player 1	U	(2,3)	(4,2)
	D	(3,4)	(5,5)

Table 1: The payoffs at the state of nature s_1

		Player 2	
		L	R
Player 1	U	(4,5)	(3,4)
	D	(5,3)	(2,2)

Table 2: The payoffs at the state of nature s_2

the usual (see e.g. Osborne and Rubinstein's textbook [22]) and the above rationalizability concepts differ, both catch the same intuition of rationalizability, and the only reason for introducing a new terminology is that this "different" concept of rationalizability reflects the main massage of the example more and makes us possible to keep the discussion quite simple.

It is easy to verify that in this example for both players both actions are rationalizable. If player 1 believes with probability 1 that the state of the nature is s_1 then her rationality implies that she plays action D. Furthermore, if player 1 believes with probability 1 that the state of the nature is s_2 and that player 2 believes with probability 1 that the state of the nature is s_1 and that she (player 1) also believes with probability 1 that the state of the nature is s_1, then that she believes that player 2 is rational, and that player 2 believes that she is rational imply that she believes with probability 1 that player 2 believes with probability 1 that she plays action D, hence she believes with probability 1 that player 2 plays action R, therefore she plays action U.

If player 2 believes with probability 1 that the state of the nature is s_2 then her rationality implies that she plays action L. If it is mutually believed with probability 1 that the state of the nature is s_1 then that player 2 believes that player 1 is rational implies that she believes with probability 1 that player 1 plays action D, hence she plays action R.

Summing up the above discussion, an adequate type space must reflect the fact that for both players both actions are rationalizable. If this does not happen then the given type space is inappropriate for modeling the incomplete information situation under consideration.

In the following we look into the question of what kind of type spaces can be appropriate for the modeling proposes under discussion.

Case 1: The type space is neither complete nor universal.

Consider the type space (see Definition 4)

$$(S, (\Omega, \mathcal{M}_i)_{i=0,1,2}, g, \{f_i\}_{i=1,2}) \ , \qquad (1)$$

where $\Omega \doteq S \times \{t_1\} \times \{t_2\}$, $f_1 \doteq \delta_{(s_2,t_2)}$, $f_2 \doteq \delta_{(s_2,t_1)}$ (the Dirac measures concentrated at point $(s_2,)$ and (s_2, t_1) respectively), $g : \Omega \to S$ is the coordinate projection, $\mathcal{M}_i \doteq \mathcal{P}(\Omega)$ (the class of all subsets of Ω) $i = 0, 1, 2$.

It is easy to verify that in model (1) at every state of the world both players believe that they play the game at state of the nature s_2, hence e.g. for player 2 action R is not rationalizable.

Case 2: The type space is complete but not universal.
Consider the type space

$$(\{s_2\}, (\Omega, \mathcal{M}_i)_{i=0,1,2}, g, \{f_i\}_{i=1,2}) \ , \tag{2}$$

where $\Omega \mathrel{\hat{=}} \{s_2\} \times \{t_1\} \times \{t_2\}$, $g : \Omega \to S$ is $g \mathrel{\hat{=}} s_2$ (the natural embedding of Ω into S), $f_1 \mathrel{\hat{=}} \delta_{(s_2,t_2)}$, $f_2 \mathrel{\hat{=}} \delta_{(s_2,t_1)}$, and $\mathcal{M}_i \mathrel{\hat{=}} \{\emptyset, \Omega\}$, $i = 0, 1, 2$.

It is easy to verify that this type space is complete (see Definition 11), and at every state of the world (there is only one in this model) it is commonly believed (with probability 1) that the state of the nature is s_2, hence e.g. for player 1 action U is not rationalizable.

Case 3: Complete universal type space.
From Heifetz and Samet [15], and Meier [17]: the complete universal type space (see Definitions 7 and 11) exists. Therefore in this example it contains the type space

$$(S, (\Omega, \mathcal{M}_i)_{i=0,1,2}, g, \{f_i\}_{i=1,2}) \ , \tag{3}$$

where $T_1 \mathrel{\hat{=}} T_2 \mathrel{\hat{=}} [0, 1]$, $\Omega \mathrel{\hat{=}} S \times T_1 \times T_2$, $g : \Omega \to S$, $i = 1, 2$: $pr_i : \Omega \to T_i$ are coordinate projections, $\forall x \in [0, 1]$: $\mu(x) \in \Delta(S)$ is such that $\mu(\{s_1\}) = x$, $\mathcal{M}_0 \mathrel{\hat{=}} \sigma(\mathcal{P}(S) \otimes \{T_1\} \otimes \{T_2\})$ (σ-field generated by the sets $\mathcal{P}(S) \otimes \{T_1\} \otimes \{T_2\}$), $\mathcal{M}_1 \mathrel{\hat{=}} \sigma(\{S\} \otimes B(T_1) \otimes \{T_2\})$ ($B(T_1)$ is for the Borel σ-field of T_1), $\mathcal{M}_2 \mathrel{\hat{=}} \sigma(\{S\} \otimes \{T_1\} \otimes B(T_2))$, and last $f_1(\omega) \mathrel{\hat{=}} \mu(pr_1(\omega)) \times \delta_1$ (the product measure of the measures $\mu(pr_1(\omega))$ and δ_1), $f_2(\omega) \mathrel{\hat{=}} \mu(pr_2(\omega)) \times \delta_1$.

In this model at every state of the world every player believes that the other player believes that the state of the nature is s_1 and that the given player believes that the state of the nature is s_1, hence, as we have already discussed, for both players both actions are rationalizable.

Therefore, in this example the complete universal type space reflects the main intuitions of the modeled situation.

Case 4: Complete universal type space that does not contain every hierarchy of beliefs.

Only one question has remained, whether or not the complete universal type space contains every hierarchy of beliefs. Although in this example the universality implies that the model reflects the main intuitions of the situation we considered, in general[3], if the complete universal type space misses some hierarchies of beliefs then it is possible to construct a game in which the missing hierarchy(ies) of beliefs is(are) important, i.e., there is a game for which the complete universal type space is not appropriate (as e.g. in Case 1).

Therefore if the above mentioned failure happens then the Harsányi program breaks down, since the complete universal type space cannot reflect all important details of incomplete information situations.

[3]Brandenburger and Dekel's [7] result implies that in this very simple case every hierarchy of beliefs is in the complete universal type space. The general case is that when the parameter space S is arbitrary measurable space.

The main result of this paper (Theorem 14) argues that in the purely measurable framework Case 4 cannot happen, i.e., the complete universal type space contains every hierarchy of beliefs, in other words, the Harsányi program works.

Our result heavily depends on that we work in the purely measurable framework, i.e. with the measurable structure introduced in Definition 1. However, doing so is not restrictive at all, on the contrary, the richer structures bring only irrelevant details into the model, hence they are useless and more, as Pintér's result [24] shows, they can be harmful.

3 Type space

Notations: Let N be the set of the players, w.l.o.g. we can assume that $0 \notin N$, and let $N_0 \doteq N \cup \{0\}$, where 0 is for the nature as an extra player.

Let A be arbitrary set, then $\#A$ is for the cardinality of set A. For any $A \subseteq \mathcal{P}(X)$: $\sigma(A)$ is the coarsest σ-field which contains A. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be arbitrary measurable spaces. Then $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ or briefly $X \otimes Y$ is the measurable space on the set $X \times Y$ equipped by the σ-field $\sigma(\{A \times B \mid A \in \mathcal{M},\ B \in \mathcal{N}\})$.

The measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable isomorphic if there is such a bijection f between them that both f and f^{-1} are measurable.

Let the measurable space $(X, \mathcal{M})$ and $x \in X$ be arbitrarily fixed. Then δ_x is for the Dirac measure on $(X, \mathcal{M})$ concentrated at point x.

In the following, practically, we use the terminologies that were introduced by Heifetz and Samet [15].

Definition 1. *Let $(X, \mathcal{M})$ be arbitrarily fixed measurable space, and denote $\Delta(X, \mathcal{M})$ the set of the probability measures on it. Then the σ-field $\mathcal{A}^*$ on $\Delta(X, \mathcal{M})$ is defined as follows:*

$$\mathcal{A}^* \doteq \sigma(\{\{\mu \in \Delta(X, \mathcal{M}) \mid \mu(A) \geq p\},\ A \in \mathcal{M},\ p \in [0, 1]\}) \ .$$

In other words, $\mathcal{A}^*$ is the smallest σ-field among the σ-fields which contain the sets $\{\mu \in \Delta(X, \mathcal{M}) \mid \mu(A) \geq p\}$, where $A \in \mathcal{M}$ and $p \in [0, 1]$ are arbitrarily chosen.

In incomplete information situations it is necessary to consider the events like player i believes with probability at least p that an event occurs (beliefs operator see e.g. Aumann [2]). For this reason $\{\mu \in \Delta(X, \mathcal{M}) \mid \mu(A) \geq p\}$ must be an event (measurable set). To keep the class of events as small (coarse) as possible, we use the $\mathcal{A}^*$ σ-field[4].

Notice that $\mathcal{A}^*$ is not a fixed σ-field, it depends on the measurable space on which the probability measures are defined. Therefore $\mathcal{A}^*$ is similar to the *weak** topology, which depends on the topology of the base (primal) space.

[4] For a more detailed argument see e.g. Meier [19] p. 56. "Why should the knowledge operators of the players just operate on measurable sets and not on all subsets of the space? The justification for this is that we think of events as those sets of states that the players can describe, and only those can be the objects of their reasoning. In view of this interpretation a statement saying "player i knows that the actual state of the world is in E," where E is an entity of states he cannot represent in his mind, is meaningless. Of course, it might well be that in some knowledge–belief spaces all subsets of the space of states of the world can be described by the players (for example in the finite knowledge–belief spaces), but we do not want to assume this in general."

Assumption 2. *Let* $(S, \mathcal{A})$ *be a fixed parameter space.*

Henceforth we assume that $(S, \mathcal{A})$ is the fixed parameter space that contains all states of the nature. For instance in the example of Section 2 S has two elements: $S = \{s_1, s_2\}$, and $\mathcal{A} = \mathcal{P}(S)$.

Definition 3. *Let* Ω *be the space of the states of world, and* $\forall i \in N_0$*:* $\mathcal{M}_i$ *be a* σ*-field on* Ω*. The* σ*-field* $\mathcal{M}_i$ *represents player* i*'s information,* $\mathcal{M}_0$ *is for the information available for the nature, hence it is the representative of* $\mathcal{A}$*, the* σ*-field of the parameter space* S*. Let* $\mathcal{M} \doteq \sigma(\bigcup_{i \in N_0} \mathcal{M}_i)$*, the smallest* σ*-field which contains all* σ*-fields* $\mathcal{M}_i$*.*

Every point in Ω provides a complete description of the actual state of world. It includes both the state of the nature and the players' states of mind. The different σ-fields are for modeling the informedness of the players, they have the same role as the partitions in e.g. Aumann's [1] paper have. Therefore, if $\omega, \omega' \in \Omega$ are not distinguishable [5] in the σ-field $\mathcal{M}_i$ then player i is not able to discern difference between them, i.e., she believes the same things, and behaves in the same way at the two states ω and ω'. $\mathcal{M}$ represents all information available in the model, it is the σ-field got by pooling the information of the players and the nature.

For the sake of brevity, henceforth - if it does not make confusion - we do not indicate the σ-fields. E.g. instead of $(S, \mathcal{A})$ we write S, or $\Delta(S)$ instead of $(\Delta(S, \mathcal{A}), \mathcal{A}^*)$. However, in some cases we refer to the non-written σ-field: e.g. $A \in \Delta(X, \mathcal{M})$ is a measurable set in $\mathcal{A}^*$, i.e., in the measurable space $(\Delta(X, \mathcal{M}), \mathcal{A}^*)$, but $A \subseteq \Delta(X, \mathcal{M})$ keeps its original meaning: A is a subset of $\Delta(X, \mathcal{M})$.

Definition 4. *Let* $(\Omega, \mathcal{M})$ *be the space of the states of world (see Definition 3). The type space based on the parameter space* S *is a tuple* $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{f_i\}_{i \in N})$*, where*

1. $g : \Omega \to S$ *is* $\mathcal{M}_0$*-measurable,*
2. $\forall i \in N$*:* $f_i : \Omega \to \Delta(\Omega, \mathcal{M}_{-i})$ *is* $\mathcal{M}_i$*-measurable,*

where $\mathcal{M}_{-i} \doteq \sigma(\bigcup_{j \in N_0 \setminus \{i\}} \mathcal{M}_j)$.

Put Definition 4 differently, S is the parameter space, it contains the "types" of the nature. $\mathcal{M}_i$ represents the information available for player i, hence it corresponds to the concept of type (Harsányi [12]). f_i is the type function of player i, it maps the player i's types to her (subjective) beliefs.

The above definition of type space differs from Heifetz and Samet's concept, but it is similar to Meier's [17], [19] type space. We do not use Cartesian product space, but refer only to the σ-fields. By following strictly Heifetz and Samet's paper, if one takes the Cartesian product of the parameter space and the type sets, and defines the σ-fields as the σ-fields induced by the coordinate projections (e.g. $\mathcal{M}_0$ is induced by the coordinate projection $pr_0 : S \times \times_{i \in N} T_i \to S$, for the notations see their paper) then she gets at our concept. However, if the Cartesian

[5] Let $(X, \mathcal{T})$ be arbitrarily fixed measurable space, and $x, y \in X$ be also arbitrarily fixed. x and y are measurably indistinguishable if $\forall A \in \mathcal{T}$: $(x \in A) \Leftrightarrow (y \in A)$.

product is not used directly then it is necessary to connect the parameter space into the type space in some way. For this we use g (Mertens and Zamir [20] and Meier [19] use a similar formalism), hence g and pr_0 have the same role in the two formalizations, in this and in Heifetz and Samet's paper respectively.

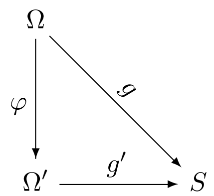
A further difference between the two formalizations lies in the role of the parameter space. While in Heifetz and Samet's paper the entire parameter space must appear in the type space, in our approach that is not required (see Case 2 in the example of Section 2). We emphasize that this difference is not relevant.

Definition 5. *The type morphism between the type spaces*

$$(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{f_i\}_{i \in N}) \quad \textit{and} \quad (S, \{(\Omega', \mathcal{M}'_i)\}_{i \in N_0}, g', \{f'_i\}_{i \in N})$$

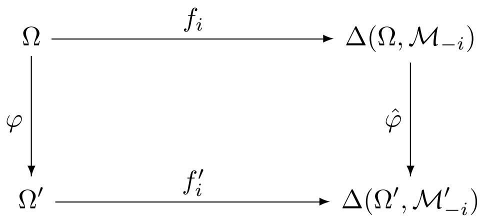
$\varphi : \Omega \to \Omega'$ *is such an* $\mathcal{M}$*-measurable function that*

1. *diagram* (4) *is commutative, i.e.* $\forall \omega \in \Omega$: $g(\omega) = g' \circ \varphi(\omega)$,



(4)

2. $\forall i \in N$: *diagram* (5) *is commutative, i.e.* $\forall i \in N$, $\forall \omega \in \Omega$: $f'_i \circ \varphi(\omega) = \hat{\varphi} \circ f_i(\omega)$,



(5)

where $\hat{\varphi} : \Delta(\Omega, \mathcal{M}_{-i}) \to \Delta(\Omega', \mathcal{M}'_{-i})$ *is defined as follows:* $\forall \mu \in \Delta(\Omega, \mathcal{M}_{-i})$, $\forall A \in \mathcal{M}'_{-i}$: $\hat{\varphi}(\mu)(A) = \mu(\varphi^{-1}(A))$. *It is a slight calculation to see that* $\hat{\varphi}$ *is a measurable mapping.*

φ *type morphism is a type isomorphism, if* φ *is a bijection and* φ^{-1} *is also a type morphism.*

The above definition is practically the same as Heifetz and Samet's, hence all intuitions, they discussed, remain valid, i.e., the type morphism maps type profiles from a type space to type profiles in an other type space in a way that the corresponded types induce equivalent beliefs. In other words, the type morphism reserves the players' beliefs.

Corollary 6. *The type spaces that are based on the parameter space* S *as objects and the type morphisms form a category. Let* $\mathcal{C}^S$ *denote this category of type spaces.*

Proof. It is a direct corollary of Definitions 4 and 5. Q.E.D.

Heifetz and Samet introduced the concept of universal type space.

Definition 7. *The type space* $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{f_i\}_{i \in N})$ *is universal, if for any type space* $(S, \{(\Omega', \mathcal{M}'_i)\}_{i \in N_0}, g', \{f'_i\}_{i \in N})$ *there is a unique type morphism* φ

$$\textit{from} \quad (S, \{(\Omega', \mathcal{M}'_i)\}_{i \in N_0}, g', \{f'_i\}_{i \in N}) \quad \textit{to} \quad (S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{f_i\}_{i \in N}) \ .$$

In other words, the universal type space is the most general, the broadest type space among the type spaces. It contains all types that appear in the type spaces of the given category.

Corollary 8. *The universal type space is terminal (final) object in* $\mathcal{C}^S$*.*

Proof. It comes directly from Definition 7. Q.E.D.

From the viewpoint of category theory the uniqueness of universal type space is really straightforward.

Corollary 9. *The universal type space is unique up to type isomorphism.*

Proof. Every terminal object is unique up to isomorphism. Q.E.D.

The only question is the existence of universal type space.

Proposition 10. *There is universal type space, in other words, there is terminal object in* $\mathcal{C}^S$*.*

Proof. See Heifetz and Samet Theorem 3.4. Q.E.D.

As we have already mentioned, Heifetz and Samet's formalization of type space is a little bit different from ours. However the difference between the two approaches is quite slight, and we prove a stronger result in Theorem 14, hence we have omitted the formal proof of the above proposition.

Next, we turn our attention to an other property of type spaces, the completeness.

Definition 11. *The type space* $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N_0}, g, \{f_i\}_{i \in N})$ *is complete if* $\forall i \in N$*:* f_i *is surjective (onto).*

The above concept was introduced by Brandenburger [5]. Completeness recommends that for any player i, every probability measure on $(\Omega, \mathcal{M}_{-i})$ be in the range of the given player's type function. In other words, for any player i: all measures on $(\Omega, \mathcal{M}_{-i})$ must belong to types of the given player.

Proposition 12. *The universal type space is complete.*

Proof. See Meier [17] Theorem 4. Q.E.D.

We can say again that Meier's type space is a little bit different from ours, however the difference is really slight, and we prove a stronger result in Theorem 14, hence we have omitted the formal proof of the above proposition.

4 Beliefs space

In the following we formalize the intuition of hierarchy of beliefs, i.e., the "infinite regress in reciprocal expectations." First we give a rough description (see Mertens and Zamir's [20]).

Take player i, and examine the situation from her viewpoint:

$$\begin{aligned}
T_0 &\doteq S \\
T_1 &\doteq T_0 \otimes \Delta(T_0)^{N\setminus\{i\}} \\
T_2 &\doteq T_1 \otimes \Delta(T_1)^{N\setminus\{i\}} = T_0 \otimes \Delta(T_0)^{N\setminus\{i\}} \otimes \Delta(T_0 \otimes \Delta(T_0)^{N\setminus\{i\}})^{N\setminus\{i\}} \\
&\vdots \\
T_n &\doteq T_{n-1} \otimes \Delta(T_{n-1})^{N\setminus\{i\}} = T_0 \otimes \bigotimes_{m=0}^{n-1} \Delta(T_m)^{N\setminus\{i\}} \\
&= T_0 \otimes \bigotimes_{m=0}^{n-2} \Delta(T_m)^{N\setminus\{i\}} \otimes \Delta(T_0 \otimes \bigotimes_{m=0}^{n-2} \Delta(T_m)^{N\setminus\{i\}})^{N\setminus\{i\}} \\
&\vdots
\end{aligned}$$

The above formalism can be interpreted as follows. T_0 describes the basic uncertainty of the modeled situation, it consists of the states of the nature. T_1 is for T_0 and the first order beliefs of the other players (not i) $\Delta(T_0)^{N\setminus\{i\}}$ ($N \setminus \{i\}$ is the players' set except player i), i.e., what the other players believe about the states of the nature. In general, T_n describes T_{n-1} and the nth order beliefs of the other payers $\Delta(T_{n-1})^{N\setminus\{i\}}$, i.e., what the other players believe about T_{n-1}.

However, there is some redundancy[6] in the above description. E.g. $\Delta(T_0 \otimes \Delta(T_0)^{N\setminus\{i\}})^{N\setminus\{i\}}$ determines $\Delta(T_0)^{N\setminus\{i\}}$ and so does $\Delta(T_{n-1})^{N\setminus\{i\}}$ $\forall(0 \leq m \leq n-2)$: $\Delta(T_m)^{N\setminus\{i\}}$, therefore we can rewrite the above formalisms into the following form:

$$\begin{aligned}
T_0 &\doteq S \\
T_1 &\doteq T_0 \otimes \Delta(T_0)^{N\setminus\{i\}} \\
T_2 &\doteq T_0 \otimes \Delta(T_0 \otimes \Delta(T_0)^{N\setminus\{i\}})^{N\setminus\{i\}} \\
T_3 &\doteq T_0 \otimes \Delta(T_0 \otimes \Delta(T_0)^{N\setminus\{i\}} \otimes \Delta(T_1)^{N\setminus\{i\}})^{N\setminus\{i\}} \\
&\vdots \\
T_n &\doteq T_0 \otimes \Delta(T_0 \otimes \bigotimes_{m=0}^{n-2} \Delta(T_m)^{N\setminus\{i\}})^{N\setminus\{i\}} \\
&\vdots
\end{aligned} \tag{6}$$

Let $\#\Theta^i_{-1} = 1$, and $\forall n \in \mathbb{N}$: let $\Theta^i_n \doteq \Delta(T_n)$, $q^i_{-10} : \Theta^i_0 \to \Theta^i_{-1}$. Moreover, $\forall n \in \mathbb{N}$, and $\forall \mu \in \Theta^i_{n+1} \doteq \Delta(T_0 \otimes \Delta(T_0 \otimes \bigotimes_{m=0}^{n-1} \Delta(T_m)^{N\setminus\{i\}}))$: μ can be naturally defined on $T_0 \otimes \Delta(T_0 \otimes \bigotimes_{m=0}^{n-2} \Delta(T_m)^{N\setminus\{i\}})$ as a restriction of μ, i.e. let $q^i_{nn+1} : \Theta^i_{n+1} \to \Theta^i_n$ be as follows $\forall \mu \in \Theta^i_{n+1}$:

$$q^i_{nn+1}(\mu) \doteq \mu|_{T_0 \otimes \Delta(T_0 \otimes \bigotimes_{m=0}^{n-1} \Delta(T_m)^{N\setminus\{i\}})} .$$

[6] This redundancy is called *coherency* and *consistency* in the literature of game theory and mathematics respectively.

Then $\forall n$: q^i_{nn+1} is a measurable mapping.

The next definition, which is from Mertens et al.'s [21], summarizes the above discussions and formalizes the "infinite regress in reciprocal expectations."

Definition 13. *In the diagram* (7)

$$\begin{array}{ccc}
\Theta^i & & \Delta(S \otimes \Theta^{N\setminus\{i\}}) \\
\Big\downarrow p^i_{n+1} & & \Big\downarrow id_S \quad \Big\downarrow p^{N\setminus\{i\}}_n \\
\Theta^i_{n+1} & \doteq & \Delta(S \otimes \Theta^{N\setminus\{i\}}_n) \\
\Big\downarrow q^i_{nn+1} & & \Big\downarrow id_S \quad \Big\downarrow q^{N\setminus\{i\}}_{n-1n} \\
\Theta^i_n & \doteq & \Delta(S \otimes \Theta^{N\setminus\{i\}}_{n-1})
\end{array} \tag{7}$$

- $i \in N$ *is an arbitrarily fixed player,*
- $n \in \mathbb{N}$*,*
- S *is the fixed parameter space,*

  moreover $\forall i \in N$*:*
- $\#\Theta^i_{-1} = 1$*,*
- $\forall n \in \mathbb{N} \cup \{-1\}$*:* $\Theta^{N\setminus\{i\}}_n \doteq \bigotimes_{j \in N\setminus\{i\}} \Theta^j_n$*,*
- $q^i_{-10} : \Theta^i_0 \to \Theta^i_{-1}$*,*
- $\forall m, n \in \mathbb{N}$*,* $m \leq n$*,* $\forall \mu \in \Theta^i_n$*:*

$$q^i_{mn}(\mu) \doteq \mu|_{S\otimes\Theta^{N\setminus\{i\}}_{m-1}} ,$$

  therefore q^i_{mn} *is a measurable mapping.*
- $\Theta^i \doteq \varprojlim(\Theta^i_n, \mathbb{N} \cup \{-1\}, q^i_{mn})$*,*
- $\forall n \in \mathbb{N} \cup \{-1\}$*:* $p^i_n : \Theta^i \to \Theta^i_n$ *is canonical projection,*
- $\forall m, n \in \mathbb{N} \cup \{-1\}$*,* $m \leq n$*:* $q^{N\setminus\{i\}}_{mn}$ *is the product of the mappings* q^j_{mn}*,* $j \in N \setminus \{i\}$*, and so is* $p^{N\setminus\{i\}}_n$ *of* p^j_n*,* $j \in N \setminus \{i\}$*, therefore both mappings are measurable,*
- $\Theta^{N\setminus\{i\}} \doteq \bigotimes_{j \in N\setminus\{i\}} \Theta^j$*.*

Then $T \doteq S \otimes \Theta^N$ *is called beliefs space.*

The interpretation of the beliefs space is the following. For any $\theta^i \in \Theta^i$: $\theta^i = (\mu_1^i, \mu_2^i, \ldots)$, where $\mu_n^i \in \Theta_{n-1}^i$ is the nth order belief of player i. Therefore every point Θ^i defines an inverse system of measure spaces

$$((S \otimes \Theta_n^{N\setminus\{i\}}, p_{n+1}^i(\theta^i)), \mathbb{N} \cup \{-1\}, (id_S, q_{mn}^{N\setminus\{i\}})) \ . \tag{8}$$

We call inverse system of measure spaces like (8) player i's *hierarchy of beliefs*[7].

To sum up, T consists of all states of world: all states of the nature, the points in S, and all states of mind, the points in the set Θ^N, therefore T contains all players' all hierarchies of beliefs.

Our main result:

Theorem 14. *The complete universal type space contains all players' all hierarchies of beliefs.*

All next section is devoted to the proof of the above theorem.

5 The proof of theorem 14.

The strategy of the proof is to show that the beliefs space (see Definition 13) generates (is equivalent to) the complete universal type space (in category $\mathcal{C}^S$). The key point of the proof is to demonstrate that in the diagram (7) $\forall i \in N$:

$$\Theta^i = \Delta(S \otimes \Theta^{N\setminus\{i\}}) \ ,$$

i.e. they are measurable isomorphic. Therefore in the following we focus on this point.

Lemma 15. *The belief space T generates a type space in the category $\mathcal{C}^S$.*

Proof. Let $\forall i \in N$: $pr_i : T \to \Theta^i$, $pr_0 : T \to S$ be coordinate projections, and $\forall i \in N \cup \{0\}$: the σ-fields $\mathcal{M}_i^*$ be induced by pr_i. From Proposition 4.3 in Pintér [25] $\forall i \in N$:

$$\Theta^i = \Delta(S \otimes \Theta^{N\setminus\{i\}}) \ , \tag{9}$$

i.e., they are measurable isomorphic.

Furthermore, let $g^* \doteq pr_0$, and $\forall t \in T$: $f_i^*(t) \doteq pr_i(t)$. Then

$$(S, \{(T, \mathcal{M}_i^*)\}_{i\in N}, g^*, \{f_i^*\}_{i\in N})$$

is a type space in category $\mathcal{C}^S$. Q.E.D.

Corollary 16. *In the type space $(S, \{(T, \mathcal{M}_i^*)\}_{i\in N}, g^*, \{f_i^*\}_{i\in N})$ $\forall i \in N$: if $pr_i(t) \neq pr_i(t')$ then $f_i^*(t) \neq f_i^*(t')$.*

Proof. It is the direct corollary of that different inverse systems of measure spaces have different inverse limits, and Θ^i consists of different inverse systems of measure spaces (hierarchies of beliefs). Q.E.D.

Proposition 17. *The type space $(S, \{(T, \mathcal{M}_i^*)\}_{i\in N}, g^*, \{f_i^*\}_{i\in N})$ is complete.*

[7] In the literature this system is usually called coherent hierarchy of beliefs. Since it does not make confusion, in this paper we omit the adjective coherent.

Proof. It is the direct corollary of (9). Q.E.D.

Proposition 18. *The type space* $(S, \{(T, \mathcal{M}_i^*)\}_{i \in N}, g^*, \{f_i^*\}_{i \in N})$ *is a universal type space.*

Proof. Let $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N}, g, \{f_i\}_{i \in N})$ be an arbitrarily fixed type space (an object in $\mathcal{C}^S$), and $i \in N$ and $\omega \in \Omega$ be also arbitrarily fixed.

The first order belief of player i at state of world ω $v_i^1(\omega)$ is the measure defined as follows $\forall A \in S$:

$$v_1^i(\omega)(A) \circeq f_i(\omega)(g^{-1}(A)) \ .$$

f_i is $\mathcal{M}_i$-measurable, hence v_i^1 is also $\mathcal{M}_i$-measurable.

The second order belief of player i at state of world ω $v_2^i(\omega)$ is the measure defined as follows $\forall A \in S \otimes \Theta_0^{N \setminus \{i\}}$:

$$v_2^i(\omega)(A) \circeq f_i(\omega)((g^{-1}, (v_1^{N \setminus \{i\}})^{-1}))(A) \ ,$$

where $v_1^{N \setminus \{i\}}$ is the product of the mappings v_1^j, $j \in N \setminus \{i\}$. Since f_i is $\mathcal{M}_i$-measurable, hence v_i^2 is also $\mathcal{M}_i$-measurable.

Let $n > 1$ be arbitrarily fixed, then the nth order belief of player i at state of world ω $v_n^i(\omega)$ is the measure defined as follows $\forall A \in S \otimes \Theta_{n-2}^{N \setminus \{i\}}$:

$$v_n^i(\omega)(A) \circeq f_i(\omega)((g^{-1}, (v_{n-1}^{N \setminus \{i\}})^{-1}))(A) \ .$$

Since f_i is $\mathcal{M}_i$-measurable, hence v_n^i is also $\mathcal{M}_i$-measurable.

To sum up, there is a well defined mapping $\phi : \Omega \to S \otimes T$ as follows $\forall \omega \in \Omega$:

$$\phi(\omega) \circeq (g(\omega), (v_1^i(\omega), v_2^i(\omega), \ldots)_{i \in N}) \ . \tag{10}$$

Then it is easy to verify that

(1) ϕ is $\mathcal{M}$-measurable.

(2) The above construction implies that $\forall \omega \in \Omega$, $\forall i \in N$, $\forall A \in T$:

$$f_i^* \circ \phi(\omega)(A) = f_i(\omega)(\phi^{-1}(A)) \ ,$$

i.e., ϕ is a type morphism.

(3) From Corollary 16 ϕ is the unique type morphism from the type space $(S, \{(\Omega, \mathcal{M}_i)\}_{i \in N}, g, \{f_i\}_{i \in N})$ to $(S, \{(T, \mathcal{M}_i^*)\}_{i \in N}, g^*, \{f_i^*\}_{i \in N})$. Q.E.D.

It is worth noticing that ϕ in the above proof is not injective (one to one). If there are redundant types in a type space, i.e. such types that generate the same hierarchy of beliefs (see e.g. Ely and Peski's [10] example), then the ϕ image of redundant types is one point in the universal type space. Therefore, it is not surprising at all that there are no redundant types in the universal type space, i.e., that can be complete.

The proof of Theorem 14. From Proposition 18

$$(S, \{(T, \mathcal{M}_i)\}_{i \in N}, g^*, \{f_i^*\}_{i \in N}) \tag{11}$$

is a universal type space.

Then Corollary 9 implies that Heifetz and Samet's [15] universal type space and (11) coincide (they are type isomorphic).

From Proposition 17 (11) is complete (Meier [17] also proved this).

Finally, from Definition 13 (11) contains all hierarchies of beliefs. Q.E.D.

6 Related papers

In this section Theorem 14 is compared to the results of Heifetz and Samet [16], and Pintér [24]. These papers seem to contradict our main result, however in the following we show that it is not the case at all.

Heifetz and Samet in their paper "Coherent beliefs are not always types," as the title indicates, give an example, a hierarchy of beliefs, that can not be type in any type space. Mathematically, their counterexample is based on an exercise of Halmos's book [11], an example for an inverse system of measure spaces without inverse limit. First, we summarize their example.

Example 19. Some notations: l^* and l_* are respectively the outer and inner measures induced by the Lebesgue measure. Let $\{A_n\}_n$ be the Vitali sets from Halmos' [11] example, so it is true that $\forall n$: $A_{n+1} \subseteq A_n \subseteq [0,1]$, $l^*(A_n) = 1$, $l_*(A_n) = 0$, and $\bigcap\limits_n A_n = \emptyset$. Moreover, $\forall n$: let μ_n be the probability measures on $B(\prod\limits_{k=0}^{n} A_k)$[8] also from Halmos' example.

Look at the following inverse system of measure spaces:

$$((\prod_{k=0}^{n} A_k, B(\prod_{k=0}^{n} A_k), \mu_n), \mathbb{N}, pr_{nn+1}) \ , \tag{12}$$

where $pr_{nn+1} : \prod\limits_{k=0}^{n+1} A_k \to \prod\limits_{k=0}^{n} A_k$ is coordinate projection.

Furthermore, if $X \doteq \prod\limits_{k=0}^{n} X_k$ is a product space, and δ_x is the Dirac measure concentrated at $x \doteq (x_0, x_1, \ldots, x_n)$, then $\delta_x = \prod\limits_{k=0}^{n} \delta_{x_k}$, where δ_{x_k} is the Dirac measure concentrated at x_k.

Interpretation: There are two players, we chose one of them. Let A_0 be the parameter space (the set of the states of the nature). $A_1 \subseteq A_0$, and $\forall x \in A_1$ let x be δ_x, i.e. A_1 is[9] the set of some first order beliefs of the given player. Moreover, $A_2 \subseteq A_1$ and $\forall x \in A_2$ let x be δ_x^2, where $\delta_x^2 \doteq \delta_{\delta_x}$, i.e. $A_1 \times A_2$ is the set of some second order beliefs of the given player. In general, $\forall n \geq 3$, $\forall x \in A_n \subseteq A_{n-1}$: let x be δ_x^n, where $\delta_x^n \doteq \delta_{\delta_x^{n-1}}$, i.e. $\prod\limits_{k=1}^{n} A_k$ is the set of some n-order beliefs of the given player.

To sum up, $\forall n$: $(a_0, a_1, \ldots, a_n) \in \prod\limits_{k=0}^{n} A_k$ is $(a_0, \delta_{a_1}, \delta_{a_2}^2, \ldots, \delta_{a_n}^n) = a_0 \times \delta_{(a_1,\ldots,a_n)}$.

Put it differently, $\prod\limits_{k=0}^{\infty} A_k$ is the space of some coherent hierarchies of beliefs, therefore (12) is a hierarchy of beliefs on coherent hierarchies of beliefs. However,

[8] $B(\cdot)$ is for the Borel σ-field.

[9] Henceforth in context like this "is" means the two spaces are homeomorphic

from Halmos' example this hierarchy of beliefs is not type ((12) has no inverse limit).

Next we show that Heifetz and Samet's hierarchy of beliefs is not in the beliefs space T.

Lemma 20. (12) *is not in* T.

Proof. It is enough to show that the diagonal of $A_0 \times A_1$ is not a measurable subset of $A_0 \otimes \Delta(A_0)$. The strategy is the following: if the diagonal of $A_0 \times A_1$ is measurable subset of $A_0 \otimes \Delta(A_0)$ then $\forall B \subseteq A_0 \times \Delta(A_0)$: the intersection of the diagonal of $A_0 \otimes A_1$ and B is a measurable subset in subspace $A_0 \otimes A_0$.

Examine $A_0 \otimes A_0$, i.e. let $B \doteq A_0 \otimes A_0$ (it is clear that B is a measurable subset of $A_0 \otimes \Delta(A_0)$). Then from Example 19 $A_0 \otimes A_0$ is measurable isomorphic (actually more, it is homeomorphic) to $A_0 \otimes \Delta_D(A_0)$, where $\Delta_D(A_0)$ is for the Dirac measures on A_0. However, from the definitions of $\{A_n\}_n$ the diagonal of $A_0 \times A_1$ is not a measurable subset of (the diagonal of) $A_0 \times A_0$, hence $\mu_1 \notin \Delta(A_0 \otimes \Delta(A_0))$, where μ_1 is from Example 19. Q.E.D.

To sum up, Heifetz and Samet's counterexample is such a hierarchy of beliefs that is not among the purely measurable hierarchies of beliefs, i.e. that is not in the purely measurable beliefs space. Therefore, Heifetz and Samet's result does not contradict our result.

Quite recently, Pintér provided a negative result, he argues that there is no universal topological type space in the category of topological type spaces. Actually, this non-existence is got by topological argument, hence his negative result does not contradict this paper's positive one.

On the other hand, Pintér's result clearly shows that irrelevant details, brought in the model by topological concepts, can make real difficulties, which culminate in that the goal proving that the Harsányi program works is unreachable in the topological approach.

7 Conclusion

The main result of this paper is Theorem 14 concludes that in the purely measurable framework the Harsányi program works, i.e., the incomplete information situations can be modeled by type spaces. In this sense, this paper ends the sequence of papers focusing on the Harsányi program: Heifetz and Samet [15], and Meier [17] among others.

Theorem 14 together with Pintér's [24] result raise the problem that although in the literature mostly the topological models are popular, the purely measurable and not the topological framework is appropriate for modeling incomplete information situations. Can every result in the topological framework be translated into the purely measurable one? For this question future research can answer.

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