Scoring Rule Voting Games and Dominance Solvability*

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Abstract

This paper studies the dominance-solvability (by iterated deletion of weakly dominated strategies) of general scoring rule voting games when there are three alternatives. The scoring rules we study include Plurality rule, Approval voting, Borda rule and Relative Utilitarianism. We provide sufficient conditions for dominance solvability of general scoring rule voting games. The sufficient conditions for dominance solvability are in terms of one statistic of the game: sufficient agreement on the best alternative or on the worst alternative. We also show that the solutions coincide with the set of Condorcet Winners whenever the sufficient conditions for dominance solvability are satisfied. Approval Voting performs the best in terms of our criteria.

Keywords: Scoring Rules, Voting Games, Dominance Solvability, Iterated Weak Dominance, Condorcet Winner

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1 Introduction

One of the most fundamental problems confronting modern democracies is that of the choice of voting method to adopt to make various public decisions. In order to compare different voting rules, however, we need both a *normative* criterion, as well as a way to predict outcomes when different voting rules are used. A large literature addresses the normative criteria (see for example the recent paper by Dasgupta and Maskin (2004)). Most of this literature, however, assumes sincere (i.e. truthful) voting. In contrast, this paper examines whether and under what conditions, a non-cooperative solution concept gives rise to outcomes which satisfy some normative criteria. The normative criterion we use is that of the Condorcet Winner[1] and the non-cooperative solution concept is the iterated elimination of weakly dominated strategies.

This paper examines a class of voting rules, *scoring rules* in a decentralized setting. Scoring rules voting games consist of n agents, each agent is allowed a particular type of ballot, and then ballots are added up component wise to derive scores for various candidates. Candidates which get the highest score are elected, with ties broken by randomization. Examples of scoring rules are Plurality Rule[2] and Borda Rule[3]. In such canonical voting games, it is well known that there are multiple Nash equilibria. Indeed, when there are more than two voters, even when all voters agree on the worst candidate, there is a Nash equilibrium where they all vote for the (unanimously) worst candidate. The problem is that when a voter is not pivotal, then he is indifferent between all strategies.

In our example with two candidates, it seems implausible that a voter would vote for his worst candidate! So, one easy way to restrict strategies (and hence beliefs) is to require voting strategies to be weakly undominated.

[1] A Condorcet Winner is a candidate that beats every other candidate in pair-wise sincere voting.

[2] Voters can vote only for one candidate.

[3] Voters submit ballots that rank the three candidates.

This ensures that a candidate who is unanimously less preferred than every other would never be elected. It still need not imply that a candidate who is most preferred by every voter (hence also a Condorcet Winner) wins. A natural question then is to ask if we can do better with the iterated elimination of (weakly) dominated strategies?

If the iterated elimination of dominated strategies in the voting game leads to a unique prediction, then is the Condorcet Winner chosen? Under what conditions can we predict a unique outcome when we allow voters to eliminate dominated strategies iteratively? These are the two questions we answer in this paper. We study the application of the solution concept of iterated elimination of weakly dominated strategies to all scoring rules.

The application of iterated weak dominance has a long tradition: Farquharson (1969) was the seminal article on this topic: Iterated admissibility is called "sophisticated voting" and a voting game "determinate" if sophisticated voting led to a unique outcome. Moulin (1979) formalized these concepts in a pure game–theoretic framework. Iterated Admissibility or iterated elimination of weakly dominated strategies has been criticized by a number of authors, as a strong theoretical justification for it has been elusive. A number of recent articles however provide both learning and common knowledge justifications for it. Marx (1999) shows that when a particular type of adaptive learning process converges, then players must have learned to play strategy profiles equivalent to those that survive iterated *nice*[4] weak dominance. Rajan (1998), Gilli (2002), Battigalli and Siniscalchi (2002), Stahl (1995), Brandenburger, Friedenberg and Keisler (2008) and Ewerhart (2002) provide epistemic justifications for iterated admissibility.

Most justifications of iterated admissibility rely on a particular order of elimination: maximal simultaneous deletion for all players. In the voting games we study we show that the order of elimination we use gives the same unique outcome as maximal simultaneous deletion by pure and mixed strate-

[4]Defined in Section (2). This is also the concept employed in this paper.

gies as well. Moreover, we show that the order independence holds generically for any elimination procedure deleting weakly dominated strategies.

Our analysis focuses on the case of voting games with three candidates with complete information. It is common in the literature to compare voting systems with three alternatives; for example, Myerson and Weber (1993), Myerson (2002). Often major political elections have no more than three candidates. Moreover, for reasons that will become obvious, the number of iterations in scoring rule voting games are closely linked to the number of candidates. Hence, requiring iterated admissibility imposes stronger rationality assumptions when the number of candidates is large.

We provide sufficient conditions for all scoring rules (defined precisely in the model) to be Dominance Solvable (DS) in terms of one statistic of the game: the degree of agreement on the best or the worst alternative. It may seem implausible to ask for complete information on preferences in large elections, but in fact the informational requirements for our sufficient conditions are quite parsimonious. Indeed the assumption of complete information can be easily replaced by an assumption on voters' posterior (i.e. conditional on knowing their own types) beliefs about the others' preferences. That is, sufficient agreement can easily be expressed in terms of the prior distribution of the corresponding incomplete information game. We state our results for games with complete information only for notational convenience. When the game satisfies the sufficient conditions for Dominance Solvability, we investigate if the unique outcome is also a Condorcet Winner. We find that, whenever the sufficient conditions for the game to be DS are satisfied, the outcome coincides with the set of Condorcet Winners. This is interesting as it leads us to a normative criterion for the evaluation of different scoring rules. It turns out that Approval Voting (Brams and Fishburn (1978)) performs best in the sense that whenever the sufficient conditions for DS are satisfied for the other well known scoring rules, so are those for Approval Voting.

An attractive feature about the voting games we study is that the iterated elimination procedure takes only 2 steps of iteration and a very intuitive form. The first step, as always, relies only on rationality and knowledge of the own preferences of voters and is independent of the others' preferences. It turns out that it is also independent of the scoring rule at hand. In voting games, a weakly dominated strategy is, loosely speaking, one which weakly decreases the chances of getting a preferred candidate relative to another feasible strategy. For example in Plurality Voting, the weakly dominated strategy is the one where the worst candidate gets a vote: whenever the voter is not pivotal it does not matter which strategy he uses but whenever he is pivotal, voting for the worst candidate will do worse than voting for the highest ranked candidate. Now, whenever there is sufficient agreement on the best or worst candidate and given the common knowledge of rationality, voters know that there are some candidates who are effectively not in the race. The second step of iteration then proceeds by elimination of strategies that favor candidates who are effectively not in the race.

This sequence of iteration corresponds closely to the reasoning in voters's minds when they vote strategically. This makes such voting games eminently suitable to be used in experimental settings to test the powers of subjects with respect to iterated dominance reasoning.

The conditions seem quite strong if we make the usual assumption that voters preferences are independently and randomly chosen. However, there is no reason to assume this. For example, if there is a "polarizing" candidate who divides voters into those who love him intensely (and put him at the top of their preferences) and those who hate him intensely (and put him at the bottom of their preferences) then if the two groups are "sufficiently far" from being equal the games we study would be dominance solvable. Thus our conditions for dominance solvability will be satisfied in elections where there is (or in terms of the corresponding game of incomplete information, where the beliefs of voters are such that) either an overwhelmingly favored candidate (for some scoring rules) or an overwhelmingly un-favored candidate (for some scoring rules).

The layout of the paper is as follows. Section (2) presents the model and defines concepts and notation that will be used in the rest of the paper for the canonical voting game with three alternatives. Sections (3), (4) and (5) present a general classification of scoring rule voting games according to the sufficient conditions for DS. Section (6) compares some of the well known scoring rule voting games. We also give an example which shows that our conditions are not necessary. Section (7) concludes. The detailed proofs and the solution of the example can be found in the Appendix as well as the discussion on the assumption which ensures the order independent feature of our results.

2 The Model

There are three alternatives. The set of alternatives or candidates is denoted $X = \{x, y, z\}$ with generic elements a, b and the set of voters is N with generic element i such that $|N| = n$, where $n > 3$. This is a simplified case of the general voting game and it is the simplest case where strategic voting can occur.

A vote is a vector of real numbers from a compact set $C \subset \mathbb{R}^3$. The n votes are added componentwise to obtain scores for each of the alternatives. The chosen alternative is the one receiving the highest score, with ties broken by uniform randomization. Votes are normalized to be permutations of the vector $(0, s, 1)$, where $s \in S \subset [0, 1]$ and the set S is derived from C in the obvious way.

Scoring rules differ only in the set S. We compare five scoring rules: Plurality Rule (PR) ($S = \{0\}$), Negative Plurality Rule ($S = \{1\}$) Borda Rule (BR) ($S = \{0.5\}$), Approval Voting (AV) ($S = \{0, 1\}$), and Relative Utilitarianism (RU) ($S = [0, 1]$). PR and BR are examples of fixed budget scoring rules (S is a singleton) while AV and RU are variable budget scoring rules.[5] The social choice literature usually considers scoring rules which allow singleton sets S. Our definition of a scoring rule is more general.

The set of vote vectors V_i allowed for individual i are then given by the set of permutations of $(1, s, 0)$, where $s \in S$. We allow for abstention hence, we add the vote[6] $(0, 0, 0)$ to each V_i. For example, in PR the set of vote vectors of a voter is given by $\{(0, 0, 0)\ (1, 0, 0)\ (0, 1, 0)\ (0, 0, 1)\}$; in BR, by:

$$\left\{(0,0,0)\ \left(1,\frac{1}{2},0\right)\ \left(\frac{1}{2},1,0\right)\ \left(\frac{1}{2},0,1\right)\ \left(1,0,\frac{1}{2}\right)\ \left(0,1,\frac{1}{2}\right)\ \left(0,\frac{1}{2},1\right)\right\},$$

The profile of vote vectors is $v \in V = \prod_{i \in N} V_i$. The vote vector v_{-i} denotes the vector of votes excluding voter i.

The vector $\omega(v) = (\omega_x(v), \omega_y(v), \omega_z(v))$ denotes the component-wise sum of the vector v. That is the first, second and third component of an individual vote refers to x, y and z respectively. $\omega_a(v_{-i})$ denotes the point score for alternative $a \in X$ when individual i is excluded. The winning set of alternatives, that is, the *outcome* corresponding to strategy profile v, is denoted by $W(v)$ and consists of those alternatives that get the maximum point score:

$W(v) = \{a \in X | \omega_a(v) = \max_{b \in X}(\omega_b(v))\}$.

Whenever there is a tie then we assume a random tie breaking rule. Hence possible *outcomes* include all equi-probability lotteries between the winning candidates. Abusing notation slightly, we identify the uniform lottery over a, b with the set $\{a, b\}$. We denote

$$W \equiv \{W(v) | v \in V\} \equiv \{\{x\}, \{y\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$$

as the set of feasible winning sets, outcomes for all $v \in V$.

[5] RU is a social welfare function that consists of normalizing individual utilities between 0 and 1 and then adding them (Dhillon and Mertens, 1999).

[6] We model abstention with the vote (0,0,0), but in principle it does not affect the variability of the budget.

We assume that players have strict preferences on the set of alternatives X, which are strict orderings of X. $a \succ_i b$ denotes "a is strictly preferred to b by individual i." To make things simple we assume that preferences can be represented by a v-NM utility function u_i over lotteries on the set of alternatives. The ranking between all outcomes in W thus depends on cardinal information, e.g. whether the equi-probability lottery between x and z is strictly preferred to y by voter i. However, for the purpose of this paper it is sufficient to assume an ordering $\succ_i'$ on the set of subsets of alternatives suited to the underlying preferences over the alternatives. More precisely, instead of assuming v-NM utilities, we need only assume that whenever $x \succ_i y \succ_i z$ then $\{x\} \succ_i' \{x, y\} \succ_i' \{y\}$, $\{x\} \succ_i' \{x, z\} \succ_i' \{z\}$, $\{y\} \succ_i' \{y, z\} \succ_i' \{z\}$ and $\{x, y\} \succ_i' \{x, y, z\} \succ_i' \{y, z\}$ for all the permutations of x, y, z. For example, the relation between $\{x, z\}$ and $\{y\}$ can be arbitrary. Dealing with mixed strategies however, would again require v-NM representation of the preferences on the power set of alternatives X.

We will impose the following regularity condition (Dhillon and Lockwood, 2004) which ensures that the order of deletion of weakly dominated strategies does not matter[7]:

A1. $\quad \forall i, v, v' \; s.t. \; W(v) \neq W(v') \Rightarrow u_i(W(v)) \neq u_i(W(v'))$

This says that no player is indifferent between any two different winsets. A1 implies that no voter is indifferent between any pair of alternatives, but is stronger than that in the sense that we do not allow indifferences between any pair of equi-probability lotteries on subsets of X.

The strategic form voting game Γ is then defined by $(X, N, V, W(.), (u_i)_{i \in N})$, where V depends on the scoring rule considered. Mixed strategies are defined in the usual way.

In what follows we focus only on pure strategies that survive iterated deletion of weakly dominated strategies. Let $\emptyset \neq V_i' \subseteq V_i$ and $V' = \times_{i \in N} V_i' \subseteq V$

[7] See the discussion in Appendix A.6.

be a *restriction* of V. Any restriction of V generates a unique game given by strategy spaces V_i' and the restriction of W to V'. $\Gamma^k, k = 1, 2, ...$ denotes the reduced game after k rounds of *successive restrictions* of V and $V_i^k \subseteq V_i^{k-1}, V^k \subseteq V^{k-1}$ the corresponding strategy spaces and $W^k \equiv \{W(v)|v \in V^k\}$ denotes the corresponding set of surviving winning sets, or feasible outcomes. Let $V^0 = V$ and $\lim_{k\to\infty} V^k = \cap_{k=0}^{\infty} V^k = V^\infty$. Γ^∞ denotes the reduced game with strategy space V^∞ and W^∞ denotes the corresponding set of feasible outcomes. Abusing notation, we write $u_i(v)$ instead of $u_i(W(v))$.

Definition 1 *A mixed strategy σ_i very weakly dominates pure strategy v_i in Γ^k (or on V^k) iff $\forall v_{-i} \in V_{-i}^k$, $u_i(\sigma_i, v_{-i}) \geq u_i(v_i, v_{-i})$. σ_i weakly dominates pure strategy v_i in Γ^k iff σ_i very weakly dominates v_i in Γ^k and $\exists v_{-i}' \in V_{-i}^k$ such that $u_i(\sigma_i, v_{-i}') > u_i(v_i, v_{-i}')$.*

Definition 2 *A* full reduction of V by pure (mixed) weak dominance *is V^∞ of a successive restriction such that:*

1. *for each $k \geq 1$ and $i \in N$, for all $v_i \in V_i^{k-1} \setminus V_i^k$ there is a (mixed) $v_i' \in V_i^{k-1}$ which weakly dominates v_i on V^{k-1}*
2. *for all i and $v_i \in V_i^\infty$ there is no (mixed) strategy $v_i' \in V_i^\infty$ which weakly dominates v_i on V^∞.*

The first condition requires that the restrictions are achieved by deleting weakly dominated strategies. The second ensures that no further deletion of weakly dominated strategies is possible in V^∞. Finally, a deletion procedure is called maximal simultaneous deletion if at each step all the weakly dominated strategies of all players are deleted.

Definition 3 *A restriction is a* maximal and simultaneous reduction by pure (mixed) weak dominance *if for all player i*

$$V_i^{k-1} \backslash V_i^k = \{v_i | v_i \textit{ is weakly dominated on } V^{k-1} \textit{ by a pure (mixed) strategy}\}$$

A full reduction by pure (mixed) weak dominance is achieved by maximal simultaneous deletion *if the restrictions are maximal and simultaneous reductions by pure (mixed) weak dominance at each step* $k \geq 1$.

That is, if a full reduction by pure (mixed) weak dominance is achieved by maximal simultaneous deletion if at each step all the weakly dominated strategies of all players are deleted.

Definition 4 *The game* Γ *is said to be dominance solvable (DS) if there is a full reduction by pure weak dominance such that* W^{∞} *is a singleton.*

Notice that we define DS in terms of iterated deletion by pure weak dominance. It turns out that generically this is without loss of generality in the voting games that we consider.

In the next sections we show that most scoring rules can have very similar sufficient conditions for DS. The sufficient conditions for DS for any scoring rule can be expressed in terms of sufficient agreement on the best alternative or sufficient agreement on the worst alternative.

If there is sufficient agreement on the best or on the worst alternative, then with 2 steps of deletion we achieve full reduction and prove DS. Our restrictions turn out to be maximal and simultaneous reductions. Moreover, our assumption A1 on u_i implies that generically any full reduction (pure or mixed) gives the same unique outcome hence DS.

3 First Round Reduction

In this section, we characterize the set of weakly undominated strategies. As usual, eliminating weakly dominated strategies is independent of beliefs about others' preferences. It turns out that in scoring rule voting games the set of weakly undominated strategies does not depend on the particular scoring rule either.

The first step consists of identifying strategies that are very weakly dominated. In voting games, a very weakly dominated strategy is, loosely speaking, one which weakly decreases the chances of getting a preferred candidate relative to another feasible strategy. This idea is captured in Proposition (1) below.

Proposition 1 *Consider a voter i such that $x \succ_i y \succ_i z$. Consider two pure strategies for such a voter: $v_i, v_i' \in V_i$. Then strategy v_i very weakly dominates v_i' on any set $V' \subseteq V$ if $v_i(x) - v_i(y) \geq v_i'(x) - v_i'(y)$ and $v_i(x) - v_i(z) \geq v_i'(x) - v_i'(z)$ and $v_i(y) - v_i(z) \geq v_i'(y) - v_i'(z)$.*

Proof. Let $v = (v_i, v_{-i})$, and $v' = (v_i', v_{-i})$. Observe that $\omega_a(v) - \omega_b(v) = \omega_a(v_{-i}) - \omega_b(v_{-i}) + v_i(a) - v_i(b)$ for all $a, b \in X$. By hypothesis, $v_i(x) - v_i(y) \geq v_i'(x) - v_i'(y)$ and $v_i(x) - v_i(z) \geq v_i'(x) - v_i'(z)$ and $v_i(y) - v_i(z) \geq v_i'(y) - v_i'(z)$. So whenever $W(v') = \{x\}$, then $W(v) = \{x\}$; whenever $W(v') = \{y\}$, then $W(v) \in \{\{x\}, \{y\}, \{x, y\}\}$; whenever $W(v') = \{x, y\}$, then $W(v) \in \{\{x\}, \{x, y\}\}$; whenever $W(v') = \{x, z\}$, then $W(v) \in \{\{x\}, \{x, z\}\}$; whenever $W(v') = \{z\}$, then $W(v) \in \{\{x\}, \{z\}, \{x, y\}, \{x, z\}, \{x, y, z\}\}$; whenever $W(v') = \{y, z\}$ then $W(v) \in \{\{x\}, \{y\}, \{x, y\}, \{y, z\}\{x, y, z\}\}$ and finally whenever $W(v') = \{x, y, z\}$, then $W(v) \in \{\{x\}, \{x, y\}, \{x, y, z\}\}$. Given $i's$ preferences $\succ_i'$ strategy v_i very weakly dominates strategy v_i'. For a more detailed proof see Appendix A.1. ■

We illustrate Proposition (1) for the case of PR, $S = \{0\}$: notice that if a voter is pivotal over any set involving the worst ranked alternative, he prefers to give it zero, and if not pivotal on this alternative he loses nothing by giving it zero points. Hence, using Proposition (1), the strategy that assigns a 1 to the worst ranked alternative is very weakly dominated by the two other strategies.

The next proposition then goes on to provide a characterization of the weakly undominated set. Continuing with the example of PR, the proof works as follows: Given that there is always a profile v_{-i} in the original game for which the strategy of voting for the best ranked candidate is strictly better than voting for the worst ranked, we can show that for every voter, the strategy of voting for his worst ranked is not just very weakly dominated but is weakly dominated in the original game and can be deleted.[8]

Let $\overline{s} = \max_{s \in S}(s)$, and $\underline{s} = \min_{s \in S}(s)$. The intuition above generalizes in that with three alternatives, voters would never give less than $\overline{s}$ to their best alternative and never give more than $\underline{s}$ to their worst alternative.

Additionally, it turns out that the remaining strategies are all unique best responses to some subset of the opponents' strategies from V_{-i}. This shows that our deletion procedure is maximally simultaneous and would give the same reduction by mixed weak dominance.

Proposition 2 *The restriction $V^1 = \times_{i \in N} V_i^1$, where:*

$$V_i^1 = \{(\bar{s}, 1, 0); (1, s, 0); (1, 0, \underline{s}) | s \in S\}$$

if $x \succ_i y \succ_i z$ is a maximal and simultaneous reduction of V by pure and mixed weak dominance.

See Appendix Section A.2 for the proof of Proposition (2).

Using the first round of reduction we can then deduce the following for the second round of reduction, which we illustrate for the case of PR:

If sufficiently many voters have the same worst candidate, this means that such a candidate, say z, can never be in the winning set of candidates. Hence, even voters who have z ranked above the worst would not vote for z. This is nothing but the "wasted vote" phenomenon: voters do not vote for their first best because they fear that their first best candidate may not be in a

[8]In this particular example we do not use abstention.

close race, so that they are wasting their vote. The game is thus reduced to a game of two candidates (and two strategies in the case of PR) which is always dominance solvable. Therefore the search for sufficient conditions for dominance solvability of the PR voting game is essentially a search for conditions under which we can reduce the set of possible winning candidates[9]. This idea extends to other scoring rules as well.

Our sufficient conditions for DS revolve around finding the conditions under which we can reduce the set of possible candidates that can win the election. This could happen in two ways: either we can eliminate the candidate who is worst ranked by most voters or we can say something about the candidates who cannot lose if there is sufficient agreement on the best and then use that to reduce the set of possible candidates that can win the election. We call these two sets of sufficient conditions *Agreement on the Worst* and *Agreement on the Best* respectively. We show that if $\underline{s} \leq 1/2$, then a scoring rule voting game is DS if there is "sufficient" agreement on the worst and if $\overline{s} \geq 1/2$ then a scoring rule voting game is DS if there is "sufficient" agreement on the best. Some scoring rules like Approval Voting, Relative Utilitarianism and Borda Rule are DS when *at least one* of these conditions is satisfied while other scoring rules like Plurality Rule and Negative Plurality rule are DS when there is sufficient agreement on the worst *and* the best respectively.

The conditions for agreement on the best and agreement on the worst are not symmetric– the reason is that agreement on the best only helps us to eliminate candidates in the *Losing Set* while we are interested in reducing the possible winning outcomes. Below, we define a *Losing Set* analogously to a winning set

[9] This idea extends to the case of more than three candidates as well. It would involve deleting one candidate as a possible outcome at a time, using the generalized version of the sufficient conditions. See the generalization of PR to more than three candidates in Dhillon and Lockwood (2004).

Definition 5 $L(v) = \{a \in X | \omega_a(v) = \min_{b \in X}(\omega_b(v)\}$

We denote the set of alternatives which are Condorcet winners on X as X^{CW} Let $n_{a,b}$ denote the number of voters who prefer candidate a to candidate b. Then $X^{CW} = \{a \in X | n_{a,b} \geq \frac{n}{2}, \forall b \in X\}$ A Condorcet winner (CW) is denoted as $a^{CW} \in X^{CW}$.

We denote $N_a \subseteq N$, where $a \in X$, as the set of voters that rank a as the worst alternative. Analogously, $N'_a \subseteq N$ is the set of voters that rank a as the best alternative. We let n_a and n'_a be the number of voters in N_a and N'_a, respectively. Let $\lfloor w \rfloor$ denote the rounded down integer of the real number w while $\lceil w \rceil$ refers to the rounded up integer value of w.

Denote the budget $B(v) = \sum_{a \in X} \omega_a(v)$. Note that $\forall v \in V$, $n(1 + \underline{s}) \leq B(v) \leq n(1 + \overline{s})$. Fixed budget rules are therefore those for which S is a singleton, so that $\forall v, B(v) = n(1 + s)$, while variable budget rules are those for which $B(v)$ depends non-trivially[10] on v.

4 Agreement on the worst candidate

In this section we assume, without any loss of generality, that z is the candidate that most voters rank worst[11]. Recall that n_a is the number of voters who rank a worst. So we assume w.l.o.g that $n_z = \max(n_x, n_y, n_z)$.

In the next theorem we derive sufficient conditions for Dominance Solvability of Scoring Rule voting games with $\underline{s} \leq 1/2$. We know from Proposition (2) that voters will give at most $\underline{s}$ to their worst candidate in the reduced game after one round of iterated elimination. This implies that z can get at most $n_z\underline{s} + (n - n_z)$. It is quite intuitive that if there are sufficiently many

[10]We do not consider variations due to abstention as it is weakly dominated.

[11]This is uniquely defined if z is a Condorcet Loser. Whenever our sufficient conditions are satisfied, this is indeed the case.

voters who rank z worst then z can never be in the winning set. Then, given that z cannot be in the winning set for any profile, even the voters who do not have z as the worst candidate will not waste their votes on z: It is then possible to reduce this game to one where each player wants to maximally differentiate between x and y only. Since the surviving strategies are sincere between x and y, the CW must win. Feasibility requires $n \geq n_z$ and hence that $\underline{s} \leq \frac{1}{2}$. So only scoring rules with $\underline{s} \leq \frac{1}{2}$ can satisfy this condition. This is what we show in the following result.

Theorem 1 *Agreement on the worst candidate:*

(A) *if* $n_z > \frac{n(2-\underline{s})}{3(1-\underline{s})}$, *then the game* Γ *is DS.*

(B) *if* $\underline{s} = \frac{1}{2}$ *and* $n_z = n$ *then if either (i)* n *is odd or (ii)* n *is even and* $n'_x \neq n'_y$ *then the game* Γ *is DS.*

Moreover, whenever the sufficient conditions (A) or (B) are satisfied,

1. $W^2 = W^\infty \subset \{\{x\},\{y\},\{x,y\}\}$ *full reduction is achieved by maximal simultaneous deletion by pure and mixed weak dominance. Moreover* $|W^2| = 1$ *that is the game is DS.*
2. *if* n *is odd, a unique CW,* $a^{CW} \in \{x,y\}$ *exists and* $W^\infty = \{a^{CW}\}$*; if* n *is even, at least one CW exists and* $W^\infty = X^{CW}$.

The proof can be found in the Appendix A.3. There we show that under condition (A) or (B) z cannot be a part of the winning set. At this point we know that $W^1 \subseteq \{\{x\},\{y\},\{x,y\}\}$. Next we shows that each voter has only one strategy that very weakly dominates the others. Whenever a very weakly dominant strategy is also weakly dominant there is no problem and that voter will have a unique weakly dominant strategy. The only problem is if a voter has strategies that are redundant[12]: in this case we do not know

[12]See the exact Definition 6 in the Appendix A.3.

which of the two to eliminate. We show that if a strategy is redundant then this strategy can be eliminated without changing the winset. Then the proof of Theorem 1 simply follows, from these observations.

5 Agreement on the best candidate:

Here we assume, without loss of generality, that x is the alternative that a plurality of voters rank best (this is always unique when our sufficient conditions are satisfied). As shown by Proposition (2) the reduced game Γ^1 gives a maximum of $\underline{s}$ to the worst alternative and a minimum of $\overline{s}$ to the best alternative. Thus $\min_{v \in V^1} \omega_x(v) = n'_x \overline{s}$.

Recall $n(1+\underline{s}) \leq B(v) \leq n(1+\overline{s})$, for all v. Recall too that the strategies that survive in Γ^1 for i such that $x \succ_i y \succ_i z$ are of the form $(1, 0, \underline{s})$; $(1, s, 0)$; $(\overline{s}, 1, 0)$ (see Proposition (2)).

When a sufficiently large number of voters agree that x is the best candidate, then we might conjecture that x is always in the winning set, and the only question is whether it is uniquely in the winning set or which other candidates tie with x. The next result shows conditions under which x is the *unique* candidate in the winning set. The conditions for DS are not symmetric with those where $\underline{s} \leq \frac{1}{2}$. This is because the first set of conditions (Agreement on the worst) works (directly) through ensuring that a candidate is never in the winning set. The conditions based on Agreement on the best candidate, however work indirectly by first proving that the candidate who is top ranked by a sufficiently large number of voters ($n'_x > \frac{n(1+\overline{s})}{3\overline{s}}$) can never be in the losing set – this need not imply that such a candidate is always in the winning set. The condition $n'_x > \frac{n(1+\overline{s}-\underline{s})}{2+\overline{s}-2\underline{s}}$ guarantees that x is always the unique winner. Hence the sufficient condition for DS is that $n'_x > max\left[\frac{n(1+\overline{s})}{3\overline{s}}, \frac{n(1+\overline{s}-\underline{s})}{2+\overline{s}-2\underline{s}}\right]$. Feasibility requires that $n \geq n'_x$ and this implies both $\overline{s} \geq \frac{1}{2}$, and $\underline{s} < 1$. Hence only scoring rules with $\overline{s} \geq \frac{1}{2}$ and $\underline{s} < 1$ can satisfy the conditions for agreement on the best.

The logic of the proof is illustrated quite easily with the case of Approval Voting, where $\overline{s} = 1$, and $\underline{s} = 0$ hence the sufficient conditions are satisfied if $n'_x > \frac{2n}{3}$.

By Proposition (2), the surviving strategies for a voter $i \in N'_x$ such that $x \succ_i y \succ_i z$ are $(1, 0, 0)$ and $(1, 1, 0)$. If $n'_x > \frac{2n}{3}$, then x is never in the losing set. This implies that whenever a voter $i \in N'_x$ is pivotal he is pivotal only on subsets including x and in this case he prefers to use $(1, 0, 0)$.If he is never pivotal then the strategy $(1, 1, 0)$ is redundant and he can choose $(1, 0, 0)$. Thus, $n'_x > \frac{2n}{3}$ of the voters vote 1 for x and 0 to other candidates, hence x is the unique winner.

We generalize this logic below: Denote $n'_T = \frac{n(1+\bar{s})}{3\bar{s}}$ and $n'_t = \frac{n(1+\bar{s}-\underline{s})}{2+\bar{s}-2\underline{s}}$

Theorem 2 *Agreement on the best candidate*

(A) *If* $n'_x > \max(n'_T, n'_t)$ *then the game* Γ *is DS*

(B) *Let* $\bar{s} = \frac{1}{2}$, $\underline{s} < 1$ *and* $n'_x = n$, *and either (i)* n *is odd or (ii)* n *is even and* $n_y \neq n_z$ *then the game* Γ *is DS*

Moreover, if the sufficient conditions (A) or (B) are satisfied, then

1. $W^2 = W^\infty$ *full reduction is achieved by maximal simultaneous deletion by pure and mixed weak dominance.*

2. $X^{CW} = \{x\} = W^\infty$.

The proof can be found in Appendix A.3, where first we show that under conditions (A) or (B) x can never be in the losing set, not even as a tie. This implies that the winning set W^1 consists of $\{\{x\}, \{x, y\}, \{y\}, \{x, z\}, \{z\}\}$. Then we show that there is one very weakly dominant strategy. Finally, deleting redundant strategies completes the proof.

6 Results for scoring rules

What can we say about sufficient conditions for DS of the scoring rules that are familiar in the literature? Our first result is that the sufficient conditions for DS for both AV and RU are exactly the same! Thus, there is no loss in restricting strategies to be $s \in \{0, 1\}$. All the corollaries in this section follow from the results in Sections (4) and (5).

Corollary 3 *The AV and RU voting games are DS if more than two third of the voters agree on the worst or on the best alternative. If the sufficient conditions for DS are satisfied, there exists a CW and the winning set coincides with the set of CW's.*

Corollary 4 *The BR game is DS if either there is full agreement on the best and there is no tie on the worst or there is full agreement on the worst and no tie on the best alternative. If these sufficient conditions are satisfied, in all cases there is a unique CW and it is the unique winning alternative.*

While NPR is DS under some conditions, the sufficient conditions are stronger and we do not describe them here: this is because $n'_t = n$ for NPR so it fails the proof at the second round (Lemma 11). The intuition is that in the case of NPR even if all voters have the same preferences, $x \succ_i y \succ_i z$ the only dominated strategy is $(0, 1, 1)$. Say that there are 5 voters, and consider voter 1: if all others use $(1, 1, 0)$ then to break the tie the strategy $(1, 0, 1)$ is a UBR and similarly if all others use $(1, 0, 1)$ then to break the tie, $(1, 1, 0)$ is a UBR and so none of these can be eliminated.

Define a generalized scoring rule (GSR) as one which has $S = \{s\}$, where $0 \leq s \leq 1$. The special cases are: $s = \frac{1}{2}$ is the BR, $s = 0$ is the PR and $s = 1$ is the NPR. In this case $\overline{s} = \underline{s} = s$, and so $n'_T = \frac{n(1+s)}{3s}$ and $n'_t = \frac{n}{2-s}$. So, if $s > \frac{1}{2}$ we have that $n'_T \geq n'_t$ iff $s \leq 0.7$.

Corollary 5 *Suppose $s > \frac{1}{2}$. The GSR is DS if there is sufficient agreement on the best i.e. $n'_x > \frac{n(1+s)}{3s}$ whenever $s \in (\frac{1}{2}, 0.7]$ and $n'_x > \frac{n}{2-s}$ whenever $s \in [0.7, 1)$. If the sufficient conditions are satisfied, there exists a unique CW which is the unique outcome. Suppose $s < \frac{1}{2}$ The GSR is DS if there is sufficient agreement on the worst, i.e. $n_z > \frac{n(2-s)}{3(1-s)}$. If the sufficient conditions are satisfied, there exists a CW and the winning set coincides with the CW.*

We show below how the different scoring rules fit given the two types of sufficient conditions where the GSR's fit on the 45 degree line where $\overline{s} = \underline{s} = s$:

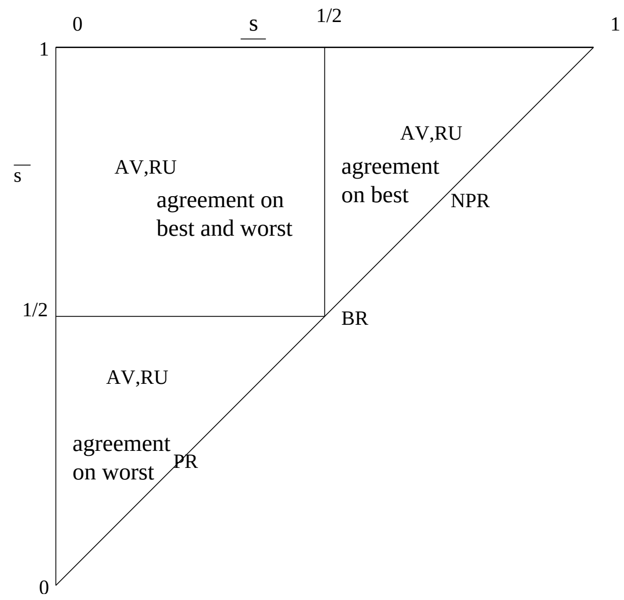

Figure 1: Scoring rules and the sufficient conditions

Given that whenever our sufficient conditions are satisfied, the Condorcet winner is chosen, we might use the strength of the sufficient conditions as a criterion on the basis of which we choose scoring rules. Thus, scoring rules with sufficient conditions that are easier to satisfy should be "better" as they choose the Condorcet winner more often with this solution concept. By this criterion, RU and AV far the best. We therefore support AV as it is a lot simpler than RU to implement and are equivalent in terms of the sufficient conditions. This is what we show in the next Corollary:

Corollary 6 *Whenever the sufficient conditions for Dominance Solvability for GSR are satisfied, so are those for RU and AV.*

We cannot say whether the sufficient conditions derived above are also necessary[13]. It is easier to try to prove necessity by deriving sufficient conditions for non-DS that rely only on ordinal information. Dhillon and Lockwood (2004) follow precisely this approach in the case of PR. The approach is essentially based on showing that each of the surviving strategies after the first round (Γ^1) are unique best responses (UBR) to some profile, and the sufficient conditions guarantee that such profiles exist.

However, the sufficient conditions for non Dominance Solvability for AV, BR and RU are very complicated to characterize. We conjecture that they require more information than just agreement on the best or worst candidate. Below we construct an example to show that the sufficient conditions are not necessary in the case of AV. In this example there is a unique CW and it is the unique outcome.

Example 1 (Sufficient conditions are not necessary)

Let $n = 5$ and the preferences such that:

$$\begin{aligned} 1,2,3 &: x \succ y \succ z \\ 4,5 &: y \succ z \succ x \end{aligned}$$

The unique CW is $\{x\}$. The game is DS but the sufficient conditions are not satisfied.

[13]This is because the game could be DS even if the sufficient conditions are not satisfied – recall that we wanted conditions that used information only on ordinal preferences.

We construct an example with AV that shows that when the degree of agreement is reduced from our sufficient conditions the game is not DS. The sufficient conditions for DS are stronger than for the existence of a CW. So, a CW may exist but the game may not be DS:

Example 2 (Non DS of AV)

Let $n = 7$ and the preferences such that:

$$\begin{aligned} 1,2 &: x \succ y \succ z \\ 3,4 &: x \succ z \succ y \\ 5,6 &: z \succ y \succ x \\ 7 &: y \succ z \succ x \end{aligned}$$

The unique CW is $\{x\}$. But this game is not DS, as each surviving strategy is a UBR. The proof is in Appendix A.4.

7 Conclusion

In this paper we found conditions for three-candidate voting games to be DS. These conditions are stated in terms of the largest proportion of voters who agree on which alternative is the worst (best).

Our results show that AV performs well. The intuition is that voters have much more flexibility under this rule. Ideally voters need to be able to choose to maximally differentiate between any two alternatives. BR does not allow the maximal differentiation. RU does allow it, but also allows other strategies which turn out never to be needed.

A natural question that arises at this stage is: can we say something more precise about the relations between the sufficient conditions required for Dominance Solvability of these different scoring rules? Corollary (6) allows us to compare PR, AV, and BR rules using as a criterion the conditions for the associated voting games to choose the CW as the only outcome of iterated weak dominance.

An attractive feature about the PR,AV and RU games is that the iterated elimination procedure takes a very intuitive form: we use a sequence of iterations that corresponds closely to the reasoning in voters's minds when they vote strategically: i.e. the iteration proceeds by elimination of candidates that everyone knows are effectively not in the race. Thus the steps of iterated elimination in the case of sufficient agreement on the worst correspond to reducing the set of outcomes that can occur! At every step all voters have the same strategy set up to permutations. This feature of Plurality Rule (and AV,RU) makes it eminently suitable to be used in experimental settings for testing the powers of subjects with regard to iterated dominance reasoning. In the case of sufficient agreement on the best, the iterated dominance reasoning is based on understanding that a particular candidate, x cannot lose outright, in the sense that he would never get the least votes (even if he does not win). When the sufficient conditions are satisfied, this implies that so many of the voters have x as their most preferred candidate that x is the unique winner and it does not matter what strategies are used by the voters who do not have x top ranked. A testable prediction in the case of Approval Voting would be that when the sufficient conditions are satisfied, voters should vote for x if they have x top ranked, everyone else knows their votes do not matter so they choose any strategy in V_i^1. This leads to the choice of the Condorcet Winner.

As Brams and Fishburn (2003) point out Approval Voting was adopted by a number of professional societies (e.g. the Institute of Electrical and Electronic Engineers, American Mathematical Society, American Statistical Association and some others: the Econometric Society has used AV to select fellows since 1980. On the whole the data supports the hypothesis that AV selects the Condorcet Winner more often than Plurality.

The sufficient conditions for DS do not imply, nor are implied by Single

Peakedness. The conditions seem quite strong if we make the usual assumption that voters preferences are independently and randomly chosen. However, if for example there is a "polarizing" candidate who divides voters into those who love him intensely (and put him at the top of their preference) and those who hate him intensely (and put him at the bottom of their preference), then as long as the size of the two groups is "sufficiently" far from being equal, our conditions for the DS of AV, BR and RU, will be satisfied. In general, our conditions for DS of AV, BR and RU will be satisfied in elections where there is either a overwhelmingly favored candidate or an overwhelmingly un-favored candidate.

Obviously the main part missing in this paper is necessary conditions for Dominance Solvability so that we could characterize the necessary and sufficient conditions at least for large groups of voters. We can find sufficient conditions for non-DS for PR (Dhillon and Lockwood (2004)) in terms of the largest number of voters who have the same worst alternative and NPR[14]. When the number of voters becomes sufficiently large, we can classify almost all games as DS or not DS in terms of agreement in the worst: in this sense we get necessary and sufficient conditions for DS. We conjecture that this may be difficult to do when the number of possible strategies is larger than the number of candidates. The conjecture is that conditions for non DS in other cases involve more information than agreement on the best and worst alternatives.

[14] This is available in the PhD thesis of Buenrostro and can be provided on request.

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A. Appendix

A.1 Proof of Proposition (1)

Proof. Let $v = (v_i, v_{-i})$, and $v' = (v'_i, v_{-i})$. Observe that $\omega_a(v) - \omega_b(v) = \omega_a(v_{-i}) - \omega_b(v_{-i}) + v_i(a) - v_i(b)$ for all $a, b \in X$. By hypothesis, $v_i(x) - v_i(y) \geq v'_i(x) - v'_i(y)$ and $v_i(x) - v_i(z) \geq v'_i(x) - v'_i(z)$ and $v_i(y) - v_i(z) \geq v'_i(y) - v'_i(z)$.

We can categorize the various cases according to all possible outcomes in the winset.

(i)Suppose that $W(v') = \{x\}$, then $\omega_x(v') - \omega_y(v') = \omega_x(v_{-i}) - \omega_y(v_{-i}) + v'_i(x) - v'_i(y) > 0$. By hypothesis $\omega_x(v_{-i}) - \omega_y(v_{-i})$ is the same in v, v' and $v_i(x) - v_i(y) \geq v'_i(x) - v'_i(y)$. Hence $\omega_x(v_{-i}) - \omega_y(v_{-i}) + v'_i(x) - v'_i(y) > 0$. The same logic applies for the pair x, z as well, substituting z for y in the above proof. Hence $W(v) = \{x\}$.

(ii)Suppose that $W(v') = \{x, y\}$, then $\omega_x(v') - \omega_y(v') = \omega_x(v_{-i}) - \omega_y(v_{-i}) + v'_i(x) - v'_i(y) = 0$. By hypothesis $\omega_x(v_{-i}) - \omega_y(v_{-i})$ is the same in v, v' and $v_i(x) - v_i(y) \geq v'_i(x) - v'_i(y)$. If $v_i(x) - v_i(y) = v'_i(x) - v'_i(y)$, then $\omega_x(v) - \omega_y(v) = 0$ so $W(v) = \{x, y\}$. If $v_i(x) - v_i(y) > v'_i(x) - v'_i(y)$, then $\omega_x(v) - \omega_y(v) > 0$. Also $\omega_x(v') - \omega_z(v') > 0$ implies $\omega_x(v') - \omega_z(v') > 0$ using the proof of (i), just substituting z for y. Hence $W(v) \in \{\{x\}\{x, y\}\}$.

(iii) Suppose that $W(v') = \{y\}$. $W(v') = \{y\}$ implies that $\omega_y(v') - \omega_x(v') = \omega_y(v_{-i}) - \omega_x(v_{-i}) + v'_i(y) - v'_i(x) > 0$. By hypothesis $\omega_y(v_{-i}) - \omega_x(v_{-i})$ is the same in v, v' and $v_i(y) - v_i(x) \leq v'_i(y) - v'_i(x)$. Clearly if $v_i(y) - v_i(x) = v'_i(y) - v'_i(x)$ then $W(v) = W(v') = \{y\}$. If $v_i(y) - v_i(x) < v'_i(y) - v'_i(x)$. Then $\omega_y(v') - \omega_x(v') > \omega_y(v) - \omega_x(v)$. So either, $\omega_y(v) - \omega_x(v) > 0$ or $\omega_y(v) - \omega_x(v) \leq 0$. Also, $W(v') = \{y\}$ implies that $\omega_y(v') - \omega_z(v') = \omega_y(v_{-i}) - \omega_z(v_{-i}) + v'_i(y) - v'_i(z) > 0$. Since $v_i(y) - v_i(z) \geq v'_i(y) - v'_i(z)$, this implies that $\omega_y(v) - \omega_z(v) > 0$ as well. Hence in all cases, $W(v) \in \{\{x\}, \{y\}, \{x, y\}\}$.

(iv) Suppose that $W(v') = \{y, z\}$. $W(v') = \{y, z\}$ implies that $\omega_y(v') - \omega_z(v') = \omega_y(v_{-i}) - \omega_z(v_{-i}) + v'_i(y) - v'_i(z) = 0$. If $v_i(y) - v_i(z) = v'_i(y) - v'_i(z)$ then $\omega_z(v) - \omega_y(v) = 0$. If $v_i(y) - v_i(z) > v'_i(y) - v'_i(z)$, then $\omega_y(v) - \omega_z(v) > 0$.. Also, $W(v') = \{y, z\}$ implies that $\omega_y(v') - \omega_x(v') = \omega_y(v_{-i}) - \omega_x(v_{-i}) + v'_i(y) - v'_i(x) > 0$. Since $v'_i(y) - v'_i(x) \geq v_i(y) - v_i(x)$, we have either $v'_i(y) - v'_i(x) =$

$v_i(y) - v_i(x)$, so $\omega_y(v) - \omega_x(v) > 0$ or $v_i'(y) - v_i'(x) > v_i(y) - v_i(x)$, so $\omega_y(v') - \omega_x(v') \geq \omega_y(v) - \omega_x(v)$ and in this case, we can have $\omega_y(v) - \omega_x(v) \leq 0$. Hence, $W(v) \in \{\{x\}, \{x, y\}, \{y\}, \{y, z\}, \{x, y, z\}\}$.

(v) Suppose that $W(v') = \{z\}$. Since this is the worst possible outcome for voter i, there is nothing to prove as $W(v)$ could be no worse than this.

(vi) Suppose that $W(v') = \{x, z\}$. Then $\omega_x(v') - \omega_z(v') = \omega_x(v_{-i}) - \omega_z(v_{-i}) + v_i'(x) - v_i'(z) = 0$. If $v_i(x) - v_i(z) = v_i'(x) - v_i'(z)$ then $\omega_x(v) - \omega_z(v) = 0$. If $v_i(x) - v_i(z) > v_i'(x) - v_i'(z)$, then $\omega_x(v) - \omega_z(v) > 0$. Also $W(v') = \{x, z\}$ implies that $\omega_x(v') - \omega_y(v') = \omega_x(v_{-i}) - \omega_y(v_{-i}) + v_i'(x) - v_i'(y) > 0$ so since $v_i(x) - v_i(z) \geq v_i'(x) - v_i'(z)$ implies $\omega_x(v) - \omega_y(v) > 0$. Hence $W(v) \in \{\{x, z\}, \{x\}\}$.

(vii) Suppose that $W(v') = \{x, y, z\}$. Then $\omega_a(v') - \omega_b(v') = \omega_a(v_{-i}) - \omega_b(v_{-i}) + v_i'(a) - v_i'(b) = 0$ for every pair $a, b \in X$. Since $v_i(x) - v_i(y) \geq v_i'(x) - v_i'(y)$, this implies that $\omega_x(v) - \omega_y(v) \geq 0$ and since $v_i(x) - v_i(z) \geq v_i'(x) - v_i'(z)$, this implies that $\omega_x(v) - \omega_z(v) \geq 0$. Moreover, $v_i(y) - v_i(z) \geq v_i'(y) - v_i'(z)$, so $\omega_y(v) - \omega_z(v) \geq 0$. Hence, $W(v) \in \{\{x\}, \{x, y\}, \{x, y, z\}\}$.

Given $i's$ preferences strategy v_i very weakly dominates strategy v_i'. ■

A.2 Proof of Proposition (2)

Proof.

To prove Proposition 2, we have to show that for all i having $x \succ_i y \succ_i z$ (A) the strategies in $V_i \backslash V_i^1$ are weakly dominated and (B) that the strategies in V_i^1 are all unique best responses (UBR) in V_i^1 to a subset $V_{-i}' \subset V_{-i}$ of opponents' strategies. That is part (B) shows, that for all $v_i \in V_i^1$ there is a V_{-i}' such that $\cap_{V_{-i}'} BR_i(v_{-i}) \cap V_i^1 = \{v_i\}$ where BR_i is the best response correspondence of player i. (B) implies that strategies in V_i^1 are not dominated by any pure or mixed strategy. (A) and (B) shows that V^1 is achieved by maximal simultaneous deletion by pure and also by mixed strategies.

(A)

Claim (i): $v_i' = (0, s, 1)$ is weakly dominated by $v_i = (1, s, 0)$. For all $s \in S$ $v_i' = (0, s, 1)$ is weakly dominated by strategy $v_i = (1, s, 0)$. For all $0 < s \in S$ $v_i' = (0,1,s)$ is weakly dominated by $v_i = (s,1,0)$. For all $1 > s \in S$ $v_i' = (s,0,1)$ is weakly dominated by $v_i = (1,0,s)$.

Proof: By Proposition (1) v_i very weakly dominates v_i'. Moreover v_i is a strictly better reply for $v_{-i} = ((1,0,s),(s,0,1),(0,0,0),\ldots,(0,0,0))$ than v_i' for voter i.

Claim (ii): For all $\bar{s} > s \in S$ strategies $v_i' = (s,1,0)$ is weakly dominated by $v_i = (\bar{s},1,0)$.

Proof: By Proposition (1) v_i' is very weakly dominated by v_i. Moreover v_i is strictly better reply for $v_{-i} = ((1,\bar{s},0),(0,0,0),\ldots,(0,0,0))$ than v_i' for voter i.

Claim (iii) For all $\underline{s} < s$ $v_i' = (1,0,s)$ is weakly dominated by $v_i = (1,0,\underline{s})$,

Proof: By Proposition (1) v_i' is very weakly dominated by v_i. Moreover v_i is strictly better reply for $v_{-i} = ((\underline{s},0,1),(0,0,0),\ldots,(0,0,0))$ than v_i' for voter i.

(B)

We can prove Part (B) using the following lemmas:

Lemma 1 *Let $x \succ_i y \succ_i z$. Then the strategies $(1,0,\underline{s})$, and $(\bar{s},1,0)$ are UBR in the set V_i^1 to some profile $v_{-i} \in V$.*

Proof. Let $v_i = (1,0,\underline{s})$, $v_i' = (1,s,0)$ for some $s \in S$ and $v_i'' = (\bar{s},1,0)$. Let $v = (v_i, v_{-i})$, $v' = (v_i', v_{-i})$, and $v'' = (v_i'', v_{-i})$.

First we show that v_i is a UBR in Γ^1:

Consider the following profile v_{-i}: if n is even: Let $\frac{n-2}{2}$ voters use $(1,\bar{s},0)$ and $\frac{n-2}{2}$ voters use $(\bar{s},1,0)$ and the remaining one voter use $(0,1,\underline{s})$. Then $\omega_y(v_{-i}) - \omega_x(v_{-i}) = 1$ and $\omega_x(v_{-i}) \geq 1 + \bar{s} > \omega_z(v_{-i}) = \underline{s}$.

If $s = 0$ then $v_i = v_i'$. Hence the outcome is $W(v) = \{x,y\}$ while $W(v'') = \{y\}$. (Note that since $n \geq 4$, $\omega(v)_x - \omega(v)_z \geq 2$, $\omega(v')_x - \omega(v')_z \geq 2$, $\omega(v'')_x - \omega(v'')_z \geq 1$ regardless of s). Now assume that $s > 0$. Then the outcome is $W(v) = \{x,y\}$ while $W(v') = W(v'') = \{y\}$. If n is odd: let 1 voter abstain and use the same profile as for n even, for the remaining voters.

Second we show that $v_i'' = (\bar{s}, 1, 0)$ is a UBR in Γ^1:

Consider the following profile v_{-i}: if n is even: Let $\frac{n-2}{2}$ voters use $(0, 1, \bar{s})$ and $\frac{n-2}{2}$ voters use $(0, \bar{s}, 1)$ and the remaining voter (out of $n-1$) uses $(\underline{s}, 0, 1)$. Then $\omega_z(v_{-i}) - \omega_y(v_{-i}) = 1$ and $\omega_y(v_{-i}) \geq 1 + \bar{s} > \omega_x(v_{-i}) = \underline{s}$. If $s = 1$ then $v_i' = v_i''$. Hence the outcome is $W(v'') = \{x, z\}$ while $W(v) = \{z\}$. (Note that since $n \geq 4$, $\omega(v)_y - \omega(v)_x \geq 0$, $\omega(v')_y - \omega(v')_x \geq s$, $\omega(v'')_y - \omega(v'')_x \geq 2 - \bar{s}$. It can be checked that x is not in the winning set for any of the profiles, not even as a tie). If $s < 1$ then the outcome is $W(v'') = \{y, z\}$ while $W(v) = W(v') = \{z\}$. If $n \geq 5$ is odd: let 1 voter abstain and use the same profile as for n even, for the remaining voters. Given $i's$ preferences v_i'' is a UBR.

■

Lemma 2 *Let $x \succ_i y \succ_i z$. Then the strategies $(1, s, 0)$ for $s \in S$ are UBR in the set V_i^1 to some subset of the opponents' profile $v_{-i} \in V_{-i}$.*

Proof. If S is a singleton and $s \leq 0.5$ then , $(1, s, 0)$ is a unique best response to the profile $v_{-i} = ((s, 0, 1), (0, 0, 0), \ldots, (0, 0, 0))$ giving the outcome $\{x\}$ or $\{x, z\}$. If $s > 0.5$ then is a unique best response to the profile $v_{-i} = ((s, 1, 0), (0, s, 1), (s, 0, 1), \ldots, (0, 0, 0))$, $(1, s, 0)$ giving the outcome $\{x, y\}$. Notice however, that the case of RU when $s \in \{0, 1\}$ is covered by the previous lemma. For RU, that is $S = [0, 1]$ fix $s \in (0, 1)$. For $n = 1, 2, \ldots$ let $l_n \in S$ and $r_n \in S$ such that $l_n < s < r_n$ and $lim_{n\to\infty} l_n = lim_{n\to\infty} r_n = s$. Denote with $BR(l_n) \in V_i^1$ the set of strategies which are best responses to the following profile $v_{-i} = ((0, 1, 1), (0, 1 - l_n, 1), (0, 0, 0), \ldots, (0, 0, 0))$. Denote with $BR(r_n) \in V_i^1$ the set of strategies which are best responses to the following profile $v_{-i} = ((0, 1, 0), (1, 1 - r_n, 0), (0, 0, 0), \ldots, (0, 0, 0))$. It must be that $\cap_{n=1}^{\infty}(BR(l_n) \cap BR(r_n)) = \{(1, s, 0)\}$. Thus $(1, s, 0)$ cannot be dominated on V_{-i} by any other (mixed) strategy in V_i. ■

Since the strategies in V_i^1 cannot be weakly dominated on V by another strategy in V_i^1, we can conclude that V^1 is a maximal simultaneous reduction of V by pure weak dominance. Moreover, because of the best response property of the remaining strategies, no mixed strategy can eliminate them, hence Proposition 2 follows. ■

A.3 Proof of Theorem (1)

Lemma 3 *If (A) then* $\{z\}, \{x, z\}, \{y, z\}, \{x, y, z\} \notin W^1$.

Proof. Suppose to the contrary that $W(v) = \{x, y, z\}$ for some profile v. By Proposition (2) the maximum score that z can get in any profile, is $n_z\underline{s} + (n - n_z)1$. Hence, the maximum sum of scores possible with profile v is $B(v) = 3(n_z\underline{s} + (n - n_z)1)$. However, recall that, $\forall v'$, $B(v') \geq n(1 + \underline{s})$. If $n_z > \frac{n(2-\underline{s})}{3(1-\underline{s})}$, then $B(v) < n(1 + \underline{s})$, a contradiction. This implies that $\{z\}, \{x, z\}, \{y, z\} \notin W^1$, when $\{x, y, z\} \notin W^1$, since the corresponding budget would be even smaller with any of the corresponding profiles. ■

Lemma 4 *Suppose* $\underline{s} = \frac{1}{2}$. *If* $n_z = n$ *then* $\{z\}, \{x, z\}, \{y, z\} \notin W^1$.

Proof. Suppose to the contrary that $\exists v$ such that $W(v) = \{x, z\}$: Then $\max(\omega_x(v) + \omega_z(v)) = 2(\max(\omega_z)) = 2(\frac{n}{2}) = n$. Since y is not in the winning set for v, $\max \omega_y(v) < \frac{n}{2}$. Thus, $B(v) < n(1 + \frac{1}{2}) \leq n(1 + \underline{s})$, a contradiction. The case of $\{y, z\}$ is symmetric. Clearly if there was a profile v' such that $W(v') = \{z\}$ then $B(v') < B(v) < n(1 + \frac{1}{2}) \leq n(1 + \underline{s})$. ■

Lemma 5 *If (B) then* $\{x, y, z\} \notin W^1$

Proof. Suppose that $\exists$ v such that $W(v) = \{x, y, z\}$. This implies that $\omega_x(v) = \omega_y(v) = \omega_z(v)$. Since $n_z = n$, $\max_{v \in V^1}(\omega_z(v)) = \frac{s}{2} = \frac{n}{2}$. Such a profile exists iff all $i \in N$ give $\frac{1}{2}$ to z. Since $\omega_x(v) = \omega_y(v) = \omega_z(v)$ it must be that n is even, and $\frac{n}{2}$ voters are using $(1, 0, \frac{1}{2})$ and $\frac{n}{2}$ voters are using $(0, 1, \frac{1}{2})$. This implies that $n'_x = n'_y$ since in the reduced game Γ^1, Proposition (2) shows that the surviving strategies for a voter $i \in N_z$ with $x \succ_i y$ are $(1, 0, \frac{1}{2}), (\bar{s}, 1, 0)$, and $(1, s, 0)$ while for a voter $i \in N_z$ with $y \succ_i x$ are $(0, 1, \frac{1}{2}), (1, \bar{s}, 0)$, and $(s, 1, 0)$.

Thus if n is odd, or n is even and $n'_x \neq n'_y$, no such profile exists, and $\{x, y, z\} \notin W(v)$, for any $v \in V^1$. ■

Lemma 6 *Assume* $W^1 = \{\{x\}, \{y\}, \{x, y\}\}$.

1. *Let* $x \succ_i y \succ_i z$. *Then the strategy* $v_i = (1, 0, \underline{s})$ *very weakly dominates strategies* $v'_i = (1, s, 0)$ *and* $v''_i = (\overline{s}, 1, 0)$ *on* V^1.

2. *Let* $y \succ_i x \succ_i z$.*Then the strategy* $v_i = (0, 1, \underline{s})$ *very weakly dominates strategies* $v'_i = (s, 1, 0)$ *and* $v''_i = (1, \overline{s}, 0)$ *on* V^1.

3. *Let* $z \succ_i x \succ_i y$. *Then the strategy* $v_i = (1, 0, \overline{s})$ *very weakly dominates strategies* $v'_i = (0, \underline{s}, 1)$ *and* $v''_i = (s, 0, 1)$, *on* V^1.

4. *Let* $z \succ_i y \succ_i x$. *Then the strategy* $v_i = (0, 1, \overline{s})$ *very weakly dominates strategies* $v'_i = (\underline{s}, 0, 1)$ *and* $v''_i = (0, s, 1)$.

5. *Let* $x \succ_i z \succ_i y$. *Then the strategy* $v_i = (1, 0, \underline{s})$ *very weakly dominates strategies* $v'_i = (1, \underline{s}, 0)$ *and* $v''_i = (\overline{s}, 0, 1)$ *on* V^1.

6. *Let* $y \succ_i z \succ_i x$. *Then the strategy* $v_i = (0, 1, \underline{s})$ *very weakly dominates strategies* $v'_i = (\underline{s}, 1, 0)$ *and* $v''_i = (0, \overline{s}, 1)$ *on* V^1.

Proof. The proof of parts 1,3,5 follow directly from the proof of Proposition (1) cases (i),(ii) and (iii). The proofs of parts 2,4,6 follow simply by permuting x and y in part 1. ■

Definition 6 *A strategy* v'_i *is redundant on* $V' \subseteq V$ *for* $V'' \subset V'$ *if there is* $v_i \in V''_i \setminus \{v'_i\}$ *such that for all* $v_{-i} \in V'_{-i}$ $u_i(v_i, v_{-i}) = u_i(v'_i, v_{-i})$.

Lemma 7 *Under assumption A1 if* $V' \supset V''$ *and for all* $v = (v_1, \ldots, v_n) \in V' \setminus V''$ *for all* i, v_i *is redundant on* V' *for* V'' *then* $\{W(v)|v \in V'\} = \{W(v)|v \in V''\}$.

Proof. It is clear that $\{W(v)|v \in V'\} \supseteq \{W(v)|v \in V''\}$. Take $q \in \{W(v)|v \in V'\}$ and suppose that $q \notin \{W(v)|v \in V''\}$. Thus there must be $v' = (v'_1, \ldots, v'_n) \in V' \setminus V''$ such that for all $v'' \in V'', q = W(v') \neq W(v'')$. By A1 for all i, $u_i(v') \neq u_i(v'')$ which contradicts to the fact that v'_i is redundant on V' for V''. ■

Now we can prove Theorem (1):

Proof. Suppose sufficient conditions **(A)** or **(B)** are satisfied. First we prove (1) in Theorem 1.

By Proposition (2) the set of surviving strategies in Γ^1 for voter i with preferences $x \succ_i y \succ_i z$ is $V_i^1 = \{(\bar{s}, 1, 0); (1, s, 0); (1, 0, \underline{s}) | s \in S\}$ (analogously for other voter types). Hence the game Γ^1 consists of the product of these strategy sets for each voter. By Lemma (3), (4) and (5) when the conditions (A) or (B) in Theorem 1 are satisfied, then $W^1 \subseteq \{\{x\}, \{x, y\}, \{y\}\}$. Lemma 6 shows that all voters have a strategy $v_i^\star \in V_i^1$ that very weakly dominates the others on V^1. Let $v^\star = (v_1^\star, \ldots, v_n^\star)$.

Let V^2 be a maximal simultaneous reduction of V^1 by pure weak dominance. Then all $v \in V^2 \backslash \{v^\star\}$ is redundant on V^2 for $\{v^\star\}$ and so by Lemma 7 $W^2 = W^\infty = \{W(v^\star)\}$.

The reduced game Γ^2 (after the deletion of redundant strategies) therefore consists of one strategy for each voter and hence W^2 is a singleton thus the game is DS.

For (2) in Theorem 1, observe that when the sufficient conditions (A) or (B) are satisfied, z is the unique Condorcet Loser. Hence there exists a CW, which is unique if n is odd. By the proof of (1) in Theorem (1), we know that all voters vote sincerely between x and y in the reduced game. Hence if n is odd, there is a unique CW and it is always chosen, while if n is even, then in case (A) $W^\infty = X^{CW} = \{x, y\}$. In case (B) there is always a unique CW and $W^\infty = a^{CW}$, where $a^{CW} \in \{x, y\}$.

■

A.3 Proof of Theorem (2)

Before we prove this theorem, we need a few lemmas. Recall that $n'_T = \frac{n(1+\bar{s})}{3\bar{s}}$.

Lemma 8 *If (A) then* $\{y, z\}, \{x, y, z\} \notin W^1$.

Proof. Step 1: If (A) then $\{x\}, \{x, y\}, \{x, z\}, \{x, y, z\} \notin L^1$.

Proof: Since all $i \in N'_x$ give a minimum of $\overline{s}$ in Γ^1 to x (by Proposition (2)) we have: $\min_{v \in V^1} = \omega_x(v) = n'_x \overline{s}$, for all $v \in V^1$. Suppose to the contrary that $\exists v \in V^1$ such that $\{x\} \in L(v)$. If $\{x\} \in L(v)$, $\omega_y(v) \geq \omega_x(v), \omega_z(v) \geq \omega_x(v)$. Then $\omega_x(v) + \omega_y(v) + \omega_z(v) \geq 3n'_x \overline{s} > n(1+\overline{s})$ since $n'_x > \frac{n(1+\overline{s})}{3\overline{s}}$, a contradiction since $B(v) \leq n(1+\overline{s})$. If $\exists v'$ such that $L(v') \in \{\{x,y\},\{x,z\},\{x,y,z\}\}$ then $B(v') > B(v) > n(1+\overline{s})$, a contradiction.

Step 2: Thus we can deduce that $\omega_x(v) > min(\omega_y(v), \omega_z(v))$ for all profiles $v \in V^1$.

The proof is obvious using Step 2 above: $\{y,z\}$ and $\{x,y,z\}$ cannot be in the winning set for any profile v.

■

Lemma 9 *If (B) then* $\{y,z\}, \{x,y,z\} \notin W^1$.

Proof.

Step 1: If (B) then $\{x\}, \{x,y\}, \{x,z\}\{x,y,z\} \notin L^1$. First we show that $\{x,y\}$ and $\{x,z\}$ cannot be in the losing set for any profile $v \in V^1$. Consider the case $L(v) = \{x,y\}$ (the case $L(v) = \{x,z\}$ is symmetric).

Suppose to the contrary that $L(v) = \{x,y\}$ for some $v \in V^1$. Then $\omega_x(v) = \omega_y(v) < \omega_z(v)$. Since $n'_x = n$, $\min_{v \in V^1} \omega_x(v) = \frac{n}{2}$, so that $\omega_x(v) + \omega_y(v) + \omega_z(v) > \frac{3n}{2}$, a contradiction, since $B(v) \leq \frac{3n}{2}, \forall v \in V^1$ when $\overline{s} = \frac{1}{2}$.

By the same logic if $\{x\}$ is in the losing set for some profile v, then $\omega_x(v) < \min(\omega_y(v), \omega_z(v))$. Since $n'_x = n$, $\min_{v \in V^1} \omega_x(v) = \frac{n}{2}$, we have $B(v) > \frac{3n}{2}$, a contradiction.

Suppose to the contrary that $\{x,y,z\} \in L^1$ for some profile v. Then it must be that $B(v) = \frac{3n}{2}$. The only profile in V^1 that supports this outcome is $\frac{n}{2}(\bar{s},1,0) + \frac{n}{2}(\bar{s},0,1)$. This profile exists only if n is even and $n_y = n_z$. Hence if (B) then such a profile does not exist, contradiction.

Step 2: If (B) then $\omega_x(v) > min(\omega_y(v), \omega_z(v))$ for all profiles $v \in V^1$, hence by the definition of losing set we have $W^1 \subset \{\{x\},\{y\},\{z\},\{x,y\}\{x,z\}\}$. ■

Lemma 10 *Assume* $W^1 \subseteq \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}\}$.

1. *Let* $x \succ_i y \succ_i z$. *Then the strategy* $v_i = (1, \underline{s}, 0)$ *very weakly dominates strategies* $v_i' = (1, s, 0)$ *and* $v_i'' = (\overline{s}, 1, 0)$ *on* V^1.

2. *Let* $x \succ_i z \succ_i y$. *Then the strategy* $v_i = (1, 0, \underline{s})$ *very weakly dominates strategies* $v_i' = (1, 0, s)$ *and* $v_i'' = (\overline{s}, 0, 1)$ *on* V^1.

Proof.

By Proposition (1) applied to W^1, v_i very weakly dominates v_i'.

The proof that v_i very weakly dominates v_i'' follows directly from the proof of Proposition (1), cases (i), (ii), (vi), cases (iv) and (vii) are excluded from hypothesis. It remains to show that cases (iii) and (v) still hold.

Therefore, suppose that $W(v'') = \{y\}$. Clearly $\omega_x(v) - \omega_y(v) \geq \omega_x(v') - \omega_y(v')$. By the proof of lemma (8), $\omega_x(v) > \min_{a=y,z}(\omega_a(v)), \forall v \in V^1$, i.e. x cannot be in the losing set for any profile in V^1. Hence we have the following:

$$\omega_x(v'') - \omega_z(v'') > 0 \tag{1}$$

where inequality (1) holds because in this case neither x nor y are in the losing set. The inequality (1) implies that:

$$\omega_x(v) - \omega_z(v) > 0 \tag{2}$$

This implies that z can never be in the winning set, $W(v)$, not even as a tie. Hence, $W(v) \in \{\{x\}, \{x, y\}, \{y\}\}$.

A symmetric argument holds for the case $W(v') = W(v'') = \{z\}$.

■

Comment 1: The reduced game Γ^2 is the game where *all* $i \in N_x'$ have strategies $V_i^2 \subsetneq \{(1, 0, \underline{s}); (1, \underline{s}, 0)\}$ after deleting the redundant strategies and all $j \notin N_x'$ have $V_j^2 = V_j^1$.

Lemma 11 *In the game* Γ^2 *if* $n_x' > n_t'$, *the Scoring rule game is DS and* $W^2 = \{x\}$.

Proof. First notice that $B(v') \leq n'_x(1+\underline{s})+(n-n'_x)(1+\bar{s})$ for all $v' \in V^2$. Suppose, to the contrary that there exists $v \in V^2$ such that $W(v) = \{x, y\}$. This implies that $\omega_x(v) = \omega_y(v) \geq n'_x$. Observe that then, $B(v) \geq (n - n'_x)(\underline{s}) + 2n'_x$. This is equivalent to $n'_x \leq \frac{n(1+\bar{s}-\underline{s})}{2+\bar{s}-2\underline{s}} = n'_t$ a contradiction. By the same argument, $\{x, z\}, \{y\}, \{z\} \notin W^2$.

■

Now we are ready to prove Theorem (2).

Proof.

Suppose sufficient conditions (A) or (B) are satisfied.

By Proposition (1), the set of surviving strategies in Γ^1 for voter i with $x \succ_i y \succ_i z$ are given by $V_i^1 = \{(\bar{s}, 1, 0); (1, s, 0); (1, 0, \underline{s})\}$ (analogously for other voters). Hence the game Γ^1 consists of these strategy sets for each voter. By Lemmas (8) and (9) when the conditions (A) or (B) are satisfied then $W^1 \subseteq \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}\}$. Lemma (10) shows that all voters $i \in N'_x$ have a subset of strategies $V_i^* \subset V_i^1$ such that for every strategy $v_i \in V_i^1$there exists $v_i^* \in V_i^*$ which very weakly dominates v_i. Let V^* be the product of V_i^*, $i \in N'_x$ and V_j^1 for all $j \notin N'_x$.

Let V^2 be a maximal and simultaneous reduction of V^1 by pure weak dominance. By Lemma 7 $W^2 = \{W(V^*)\}$.

Lemma (11) then shows that $W^2 = W^\infty = \{\{x\}\}$. Hence the game is DS.

■

A.4 Proof of Example 1

Proof. In Γ^1 the strategies that survive are $(1, 1, 0)$ and $(1, 0, 0)$ for 1,2 and 3; $(0, 1, 0)$ and $(0, 1, 1)$ for 4 and 5. First notice that $\{z\}, \{x, z\}, \{x, y\} \notin W^1$, since $\min_{v \in V^1}(\omega_x(v)) = 3$ and $\max_{v \in V^1}(\omega_z(v)) = 2$. So, clearly the strategy $(0, 1, 0)$ very weakly dominates strategy $(0, 1, 1)$ for voters 4,5. It also weakly dominates $(0, 1, 1)$ for voters 4,5 since there is a profile $v_{-4} = ((1, 0, 0), (1, 1, 0), (1, 1, 0), (0, 1, 0))$ in V^1 (symmetrically v_{-5}) such that $(0, 1, 0)$ is a strictly better response than $(0, 1, 1)$. Hence we can eliminate $(0, 1, 1)$

for voters 4,5. Moreover, for voters 1,2,3 the strategy $(1,0,0)$ also weakly dominates $(1,1,0)$ since it very weakly dominates it and there is a profile $v_{-1} = ((1,0,0), (1,0,0), (0,1,0), (0,1,0))$ in V^1 (symmetrically v_{-i}, for $i = 2,3$) such that $(1,0,0)$ is a strictly better response than $(1,1,0)$. Hence the game is DS and the CW is the unique outcome. ■

A.5 Proof of Example 2

Proof. In Γ^1 the strategies that survive are $(1,1,0)$ and $(1,0,0)$ for 1 and 2; $(1,0,1)$ and $(1,0,0)$ for 3 and 4; $(0,1,1)$ and $(0,0,1)$ for 5 and 6 and $(0,1,1)$ and $(0,1,0)$ for 7.

Consider voters 1 and 2: Strategy $(1,1,0)$ is a UBR to $(1,1,0)+2(1,0,1)+3(0,1,1)$, and strategy $(1,0,0))$ is a UBR to $(1,1,0)+2(1,0,0)+2(0,1,1)+(0,1,0)$. Consider voters 3 and 4: strategy $(1,0,1)$ is a UBR to $2(1,1,0)+(1,0,1)+3(0,1,1)$ and strategy $(1,0,0)$ is a UBR to $2(1,0,0)+(1,0,1)+2(0,0,1)+(0,1,1)$. Next, consider voters 5 and 6: strategy $(0,1,1)$ is a UBR to $2(1,1,0)+2(1,0,0)+(0,1,1)+(0,1,0)$ and strategy $(0,0,1)$ is a UBR to $2(1,1,0)+2(1,0,1)+2(0,1,1)$. Finally, consider 7: strategy $(0,1,1)$ is a UBR to $2(1,0,0)+(1,0,1)+(1,0,0)+2(0,0,1)$ and strategy $(0,1,0)$ is a UBR to $2(1,1,0)+2(1,0,1)+2(0,1,1)$. ■

A.6 Discussion of Order Independence

Under assumption A1 and appropriate assumptions on the set S:

1. our full reduction by pure weak dominance gives the same W^∞ for all the voting games considered in the paper as the full reduction achieved by maximal simultaneous deletion

2. the full reduction by mixed weak dominance achieved by maximal simultaneous deletion gives the same W^∞ for all the voting games considered in the paper as if it is achieved by pure weak dominance

3. for *finite* voting games (all but RU), any full reduction by pure weak dominance results in the same W^∞

4. for *generic*, *finite* voting games (all but RU), any full reduction by mixed weak dominance results in the same W^∞

The first is an immediate consequence of our construction. We show in the proofs that in each step of elimination we delete all the weakly dominated strategies or we stop when W^{∞} is a singleton.

For the second we prove, that after each step of elimination the remaining strategies are all unique best responses to some subsets of the opponents' strategies, hence could not have been removed by any mixed strategy or W^{∞} is already a singleton.

The third follows from Gretlein (1983) who assumes strict preferences over outcomes in games with *finite* strategy spaces. This is equivalent with A1. Gretlein (1983) shows that the order of pure elimination does not matter with respect to the achievable outcomes. That is W^{∞} is the same regardless of the full reduction by pure weak dominance. Thus, if we are concerned only with elimination procedures by pure weak dominance all we have to prove is that our finite voting games are dominance solvable. An alternative proof is that A1 also implies Marx and Swinkels'(1997) definition of Transference of Decisionmaker Indifference (TDI) condition[15] which is sufficient to ensure that the order of deletion does not matter if deletion happens with pure strategies in games with finite strategy spaces[16]. More precisely, as stated in Theorem 1 in Marx and Swinkels'(1997), any order of nice[17] elimination of pure strategies by pure strategies would give the same full reduction of the game (up to redundant strategies and renaming), hence by A1, the same set of outcomes[18]. However, it does not ensure that deletion of pure strategies at each stage by mixed as well as pure strategies, would give the same full

[15]Whenever $u_i(v_i, v_{-i}) = u_i(v_i', v_{-i})$ then $u_j(v_i, v_{-i}) = u_j(v_i', v_{-i})$ for all $j \in -i$.

[16]See Marx and Swinkels notes however, in footnote 10: "Dealing with infinite sequences of removals introduces issues which we would prefer not to deal with here, although the main ideas of this analysis translate directly.".

[17]Satisfying TDI for the pair of strategies involved in the dominance relation.

[18]A pure strategy v_i belonging to player i is said to be *redundant* to v_i' in Γ^k if $\forall j \in N, u_j(v_i, v_{-i}) = u_j(v_i', v_{-i}), \forall v_{-i} \in V_{-i}^k$. This does not exclude the situation that strategies are redundant in terms of payoffs yet the outcomes are different. This is why we need the stronger assumption A1, which eliminates this possibility. It is worthwhile to compare this footnote with Definition 6 and Lemma 7 in A.3 in the Appendix.

reduction of the game[19].

Concerning the fourth observation, recall Marx and Swinkels' (1997) theorem in Appendix B. It states that the closure of the set of payoff functions u_i, where the sufficient condition[20] for order independence fails when deleting with mixed strategies, has an empty interior. To put it simply, for generic, finite games the order of deletion does not matter.

[19]We could apply Proposition 2 in Marx and Swinkels's (1997). It states that any full reduction by mixed weak dominance is equivalent to a subset of any full reduction by nice mixed weak dominance. Since in our games the pure deletions are nice and the games are dominance solvable, one concludes that any full reduction by mixed weak dominance reaches the same outcome. Unfortunately Proposition 2 is not valid. It is corrected in Marx and Swinkels'(2000).

[20]The sufficient condition is that TDI (over the set of outcomes) should imply TDI* over the utility space. See the definition of TDI* therein.